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WORD PROBLEM FOR SPECIAL BRAID GROUPS [1]

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Abstract. In this paper, the main idea is to present the solvability of the word problem of pure virtual braid groups by using Gröbner-Shirshov basis theory.

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Key words: Decision problems, word problem, pure virtual braid group, Gröbner-Shirshov basis.

1. Introduction and preliminaries.

Group presentations arise in various areas of mathematics such as knot theory, topology, and geometry. Another motivation for studying group presentations is the advent of softwares for symbolic computations like *GAP*. Providing algorithms to compute group presentations of given groups (semigroups) is a great help for the developers of these softwares. So, in this work, we consider monoid presentations of pure virtual braid groups PV_n ($n \geq 3$) and find a Gröbner-Shirshov bases for these monoid presentations. Thus, by these Gröbner-Shirshov bases we characterize the structure of elements of this important group structure. Therefore, we obtain solvability of the word problem.

Algorithmic problems such as the *word, conjugacy* and *isomorphism problems* have played an important role in group theory since the work of M. Dehn in early 1900's. These problems are called *decision problems* which ask for a yes or no answer to a specific question. Among these decision problems especially the word problem has been studied widely in groups [1]. It is well known that the word problem for finitely presented groups is not solvable in general; that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group.

Virtual braid groups VB_n were introduced by L. Kauffman in [22] and studied subsequently by many authors. In fact, these groups appeared in another context under the name of n-th quasitriangular group QTr_n associated to the Yang-Baxter equations [7]. Actually the group VB_n is an extension of the classical braid group

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by the symmetric group [23]. Virtual braids have a group structure that can be described by generators and relations, generalizing the generators and relations of the classical braid group. This structure of virtual braids is worth study for its own sake. In the virtual braid group two types of crossings are allowed, firstly as usual braids or secondly as an intersection of lines on the plane. As for the classical braid groups there exists the canonical epimorphism from virtual braid groups VB_n to the symmetric group S_n with the kernel called the *pure virtual braid groups* PV_n. Here, there is a short exact sequence which gives us information about the relationship between virtual and pure virtual braid groups.

$$1 \to PV_n \to VB_n \to S_n \to 1.$$

Like the usual pure braid groups, pure virtual braid groups PV_n admit a semi-direct product decomposition [5]: for $n \geq 2$, the n-th virtual pure braid group can be decomposed as

$$PV_n = V_{n-1}^* \rtimes PV_{n-1},$$

where V_{n-1}^* is a free subgroup of PV_n and PV_1 is a trivial group.

In paper [6], authors noticed that the pure virtual braid group PV_2 is isomorphic to the free group on two generators F_2. Since the word problem for free groups is solvable, the pure virtual braid group PV_2 has solvable word problem as well. Thus, it is worth to study solvability of the word problem for pure virtual braid group PV_n $(n \geq 3)$. In [8], the authors gave a simple and easily implementable solution to the word problem for virtual braid groups.

There are many important studies about Gröbner-Shirshov bases of braid groups in literature. For instance, Bokut found Gröbner-Shirshov basis for the Braid group in the Artin-Garside generators [12]. Then, the same author studied Gröbner-Shirshov basis for the Braid group in the Birman Ko-Lee generators [13]. Later, Chen and Zhong calculated Gröbner-Shirshov bases for braid groups in Adyan-Thurston generators [18]. Hence, in this paper, by considering the importance of braid groups and their derivations we study Gröbner-Shirshov basis of pure virtual braid groups PV_n $(n \geq 3)$. As known, the method of Gröbner-Shirshov basis which is the main theme of this paper gives a new algorithm to get normal forms of elements of a group, and so a new algorithm for solving the word problem in given group.

At this part of paper we will give a short material on Gröbner-Shirshov bases.

The Gröbner basis theory for commutative algebras was introduced by Buchberger [16] and provides a solution to the reduction problem for commutative algebras. In [9], Bergman generalized the Gröbner basis theory to associative algebras by proving the "Diamond Lemma". On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [25]. In [11], Bokut noticed that Shirshov's method works for also associative algebras. Hence, for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner-Shirshov basis* theory. We may finally refer the papers [3, 4, 14, 15, 19, 20, 21, 24] for some studies over Gröbner-Shirshov bases. We can also refer the readers to a good reference [2] to understand normal forms for the monoid of positive braids by using Gröbner-Shirshov basis.

Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$. Suppose that X^* is a well ordered set. Then every nonzero polynomial $f \in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Let f and g be two monic polynomials in $K\langle X\rangle$. We then have two compositions as follows:

- If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.
- If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

If g is monic, $\overline{f} = a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then transformation $f \mapsto f - \alpha agb$ is called elimination (ELW) of the leading word of g in f.

Let $S \subseteq K\langle X\rangle$ with each $s \in S$ monic. Then the composition $(f, g)_w$ is called trivial modulo (S, w) if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K, a_i, b_i \in X^*, s_i \in S$ and $a_i\overline{s_i}b_i < w$. If this is the case, then we write $(f, g)_w \equiv 0 \ mod(S, w)$.

We call the set S endowed with the well ordering $<$ a *Gröbner-Shirshov basis* for $K\langle X \mid S\rangle$ if any composition $(f, g)_w$ of polynomials in S is trivial modulo S and corresponding w.

The following lemma was proved by Shirshov [25] for free Lie algebras with deg-lex ordering (see also [10]). In 1976, Bokut [11] specialized the Shirshov's approach to associative algebras (see also [9]). Meanwhile, for commutative polynomials, this lemma is known as the Buchberger's Theorem [16, 17].

LEMMA 1.1. (Composition-Diamond Lemma) *Let K be a field, $A = K\langle X \mid S\rangle = K\langle X\rangle/Id(S)$ and $<$ a monomial ordering on X^*, where $Id(S)$ is the ideal of $K\langle X\rangle$ generated by S. Then the following statements are equivalent:*

1. *S is a Gröbner-Shirshov basis.*
2. *$f \in Id(S) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in S$ and $a, b \in X^*$.*
3. *$Irr(S) = \{u \in X^* \mid u \neq a\overline{s}b, s \in S, a, b \in X^*\}$ is a basis for the algebra $A = K\langle X \mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process many times (maybe infinitely), we eventually obtain a Gröbner-Shirshov basis S^{comp}. We should note that such a process is called the *Shirshov algorithm.*

Throughout this paper, by considering the lengths of any two words, we will use the deg-lex ordering if the lengths of these words are different or lexicographically ordering if otherwise. Additionally the notations $(i) \wedge (j)$ and $(i) \vee (j)$ will denote the intersection and inclusion compositions of relations (i) and (j), respectively.

2. Main results. In this section, we would like to obtain solvability of the word problem for pure virtual braid groups by using Gröbner-Shirshov basis theory. Maybe, some algorithms may obtain solvability of the word problem for some algebraic structures by using computer programs, but there is no any general packet program including algorithms which show the solvability of the word problem for pure virtual braid groups. From this point of view, the results given in this paper become important. Since we use the method of Gröbner-Shirshov basis theory, presentation for pure virtual braid group PV_n $(n \geq 3)$ is required. This presentation is as follows [23]:

$$PV_n = \langle \mu_{rs}(1 \leq r \neq s \leq n) \quad ; \quad \mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}(\text{for all distinct } i,j,k), \quad \mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij} \quad (\{i,j\} \cap \{k,l\} = \emptyset)\rangle \ . \tag{1}$$

We now give the monoid presentation of PV_n given in (1) as follows:

$$\left\langle \mu_{rs}, \mu_{rs}^{-1}(1 \leq r \neq s \leq n) \quad ; \quad \mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}(\text{for all distinct } i,j,k), \quad \mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij} \quad (\{i,j\} \cap \{k,l\} = \emptyset), \quad \mu_{rs}\mu_{rs}^{-1} = \mu_{rs}^{-1}\mu_{rs} = 1 \right\rangle \ . \tag{2}$$

To obtain Gröbner-Shirshov basis of pure virtual braid group PV_n, let us order the generators as $\mu_{ij} > \mu_{ij}^{-1} > \mu_{i'j'} > \mu_{i'j'}^{-1}$ iff $i > i'$ or if $i = i'$ then $j > j'$. We note that the presentaion given in (2) will be used in our results.

Firstly, we consider the pure virtual braid group PV_3. For this group, we have generators as $\mu_{12}, \mu_{13}, \mu_{21}, \mu_{23}, \mu_{31}, \mu_{32}$. Since we have the ordering between generators given above, we can write this ordering more precisely

$$\mu_{32} > \mu_{32}^{-1} > \mu_{31} > \mu_{31}^{-1} > \mu_{23} > \mu_{23}^{-1} > \mu_{21} > \mu_{21}^{-1} > \mu_{13} > \mu_{13}^{-1} > \mu_{12} > \mu_{12}^{-1}.$$

We should note that for pure virtual braid group PV_3, the relation $\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}$ does not exist in Gröbner-Shirshov basis. Now we can give the first main result of this work.

Theorem 2.1. *A Gröbner-Shirshov basis of pure virtual braid group PV_3 consists of the following relations:*

(1) $\mu_{23}\mu_{13}\mu_{12} - \mu_{12}\mu_{13}\mu_{23}$,
(2) $\mu_{21}\mu_{23}\mu_{13} - \mu_{13}\mu_{23}\mu_{21}$,
(3) $\mu_{31}\mu_{32}\mu_{12} - \mu_{12}\mu_{32}\mu_{31}$,
(4) $\mu_{32}\mu_{12}\mu_{13} - \mu_{13}\mu_{12}\mu_{32}$,
(5) $\mu_{31}\mu_{21}\mu_{23} - \mu_{23}\mu_{21}\mu_{31}$,
(6) $\mu_{32}\mu_{31}\mu_{21} - \mu_{21}\mu_{31}\mu_{32}$,
(7) $\mu_{21}(\mu_{12}\mu_{13})^x\mu_{23} - \mu_{13}\mu_{23}\mu_{21}\mu_{12}(\mu_{13}\mu_{12})^{x-1} \quad (x \geq 1)$,
(8) $\mu_{31}(\mu_{13}\mu_{23})^x\mu_{21} - \mu_{23}\mu_{21}\mu_{31}\mu_{13}(\mu_{23}\mu_{13})^{x-1} \quad (x \geq 1)$,

(9) $\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$
$(a_1 \geq 1;\ a_2, a_3, a_4 \geq 0)$,
(10) $\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}(\mu_{21}\mu_{23})^{b_1-1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$
$(b_1 \geq 1;\ b_2, b_3, b_4 \geq 0)$,
(11) $\mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}(\mu_{12}\mu_{32})^{c_5}$
$(\mu_{23}\mu_{21})^{c_6}\mu_{31} - \mu_{12}\mu_{32}\mu_{31}\mu_{13}\mu_{31}(\mu_{21}\mu_{12})^{c_1}(\mu_{32}\mu_{12})^{c_2-1}(\mu_{21}\mu_{23})^{c_3}$
$\cdots(\mu_{21}\mu_{12})^{c_4}(\mu_{32}\mu_{12})^{c_5}(\mu_{21}\mu_{23})^{c_6} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$,
(12) $\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5}$
$(\mu_{13}\mu_{12})^{d_6}\mu_{32} - \mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}$
$\cdots(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5}(\mu_{12}\mu_{13})^{d_6} \quad (d_1, d_2 \geq 1;\ d_3, d_4, d_5, d_6 \geq 0)$,
(13) $\mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}$
$(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32} - \mu_{12}\mu_{32}\mu_{31}\mu_{13}\mu_{31}\mu_{21}(\mu_{12}\mu_{32})^{e_3}(\mu_{12}\mu_{21})^{e_2}$
$(\mu_{12}\mu_{13})^{e_1-1}(\mu_{31}\mu_{21})^{e_4-1}\cdots(\mu_{12}\mu_{13})^{e_5}(\mu_{12}\mu_{21})^{e_6}(\mu_{12}\mu_{32})^{e_7}(\mu_{31}\mu_{21})^{e_8}$
$(e_1, e_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(14) $\mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}$
$(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31} - \mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}\mu_{12}(\mu_{21}\mu_{31})^{f_3}$
$(\mu_{21}\mu_{12})^{f_2}(\mu_{21}\mu_{23})^{f_1-1}(\mu_{32}\mu_{12})^{f_4-1}\cdots(\mu_{21}\mu_{23})^{f_5}(\mu_{21}\mu_{12})^{f_6}$
$(\mu_{21}\mu_{31})^{f_7}(\mu_{32}\mu_{12})^{f_8} \quad (f_1, f_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(15) $\mu_{12}\mu_{12}^{-1} - 1$, **(16)** $\mu_{13}\mu_{13}^{-1} - 1$, **(17)** $\mu_{21}\mu_{21}^{-1} - 1$, **(18)** $\mu_{23}\mu_{23}^{-1} - 1$,
(19) $\mu_{31}\mu_{31}^{-1} - 1$, **(20)** $\mu_{32}\mu_{32}^{-1} - 1$, **(21)** $\mu_{12}^{-1}\mu_{12} - 1$, **(22)** $\mu_{13}^{-1}\mu_{13} - 1$,
(23) $\mu_{21}^{-1}\mu_{21} - 1$, **(24)** $\mu_{23}^{-1}\mu_{23} - 1$, **(25)** $\mu_{31}^{-1}\mu_{31} - 1$, **(26)** $\mu_{32}^{-1}\mu_{32} - 1$.

Proof. We need to prove that all compositions among relations (1) − (26) are trivial. To do that, we have the following ambiguties w:

(1) $\wedge$ **(15)** : $w = \mu_{23}\mu_{13}\mu_{12}\mu_{12}^{-1}$, **(2)** $\wedge$ **(1)** : $w = \mu_{21}\mu_{23}\mu_{13}\mu_{12}$,
(2) $\wedge$ **(16)** : $w = \mu_{21}\mu_{23}\mu_{13}\mu_{13}^{-1}$, **(3)** $\wedge$ **(4)** : $w = \mu_{31}\mu_{32}\mu_{12}\mu_{13}$,
(3) $\wedge$ **(15)** : $w = \mu_{31}\mu_{32}\mu_{12}\mu_{12}^{-1}$, **(4)** $\wedge$ **(16)** : $w = \mu_{32}\mu_{12}\mu_{13}\mu_{13}^{-1}$,
(5) $\wedge$ **(1)** : $w = \mu_{31}\mu_{21}\mu_{23}\mu_{13}\mu_{12}$, **(5)** $\wedge$ **(2)** : $w = \mu_{31}\mu_{21}\mu_{23}\mu_{13}$,
(5) $\wedge$ **(18)** : $w = \mu_{31}\mu_{21}\mu_{23}\mu_{23}^{-1}$, **(6)** $\wedge$ **(2)** : $w = \mu_{32}\mu_{31}\mu_{21}\mu_{23}\mu_{13}$,
(6) $\wedge$ **(5)** : $w = \mu_{32}\mu_{31}\mu_{21}\mu_{23}$, **(6)** $\wedge$ **(17)** : $w = \mu_{32}\mu_{31}\mu_{21}\mu_{21}^{-1}$,
(6) $\wedge$ **(7)** : $w = \mu_{32}\mu_{31}\mu_{21}(\mu_{12}\mu_{13})^{x}\mu_{23} \quad (x \geq 1)$,
(7) $\wedge$ **(1)** : $w = \mu_{21}(\mu_{12}\mu_{13})^{x}\mu_{23}\mu_{13}\mu_{12} \quad (x \geq 1)$,
(7) $\wedge$ **(18)** : $w = \mu_{21}(\mu_{12}\mu_{13})^{x}\mu_{23}\mu_{23}^{-1} \quad (x \geq 1)$,
(8) $\wedge$ **(2)** : $w = \mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21}\mu_{23}\mu_{13} \quad (x \geq 1)$,

(8) $\wedge$ **(7)** : $w = \mu_{31}(\mu_{13}\mu_{23})^{x_1}\mu_{21}(\mu_{12}\mu_{13})^{x_2}\mu_{23} \quad (x_1, x_2 \geq 1)$,
(8) $\wedge$ **(17)** : $w = \mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21}\mu_{21}^{-1} \quad (x \geq 1)$,
(9) $\wedge$ **(4)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}\mu_{12}\mu_{13}$
$(a_1 \geq 1;\ a_2, a_3, a_4 \geq 0)$,
(9) $\wedge$ **(6)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}\mu_{31}\mu_{21}$
$(a_1 \geq 1;\ a_2, a_3, a_4 \geq 0)$,
(9) $\wedge$ **(10)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}$
$\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$(a_1, b_1 \geq 1;\ a_2, a_3, a_4, b_2, b_3, b_4 \geq 0)$,
(9) $\wedge$ **(12)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}\mu_{23}$
$(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5}$
$(\mu_{13}\mu_{12})^{d_6}\mu_{32} \quad (a_1, d_1, d_2 \geq 1;\ a_2, a_3, a_4, d_3, d_4, d_5, d_6 \geq 0)$,
(9) $\wedge$ **(14)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}$
$(\mu_{12}\mu_{32})^{f_8}\mu_{31} \quad (a_1, f_1, f_4 \geq 1;\ a_2, a_3, a_4, f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(9) $\wedge$ **(20)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}\mu_{32}^{-1}$
$(a_1 \geq 1;\ a_2, a_3, a_4 \geq 0)$,
(9) $\wedge$ **(3)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4-1}$
$\mu_{21}\mu_{31}\mu_{32}\mu_{12} \quad (a_1, a_4 \geq 1;\ a_2, a_3 \geq 0)$,
(10) $\wedge$ **(3)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}\mu_{32}\mu_{12}$
$(b_1 \geq 1;\ b_2, b_3, b_4 \geq 0)$,
(10) $\wedge$ **(5)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}\mu_{21}\mu_{23}$
$(b_1 \geq 1;\ b_2, b_3, b_4 \geq 0)$,
(10) $\wedge$ **(8)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}$
$\mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21} \quad (b_1, x \geq 1;\ b_2, b_3, b_4 \geq 0)$,
(10) $\wedge$ **(9)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}$
$\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$(b_1, a_1 \geq 1;\ b_2, b_3, b_4, a_2, a_3, a_4 \geq 0)$,
(10) $\wedge$ **(11)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}$
$\mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}(\mu_{12}\mu_{32})^{c_5}$
$(\mu_{23}\mu_{21})^{c_6}\mu_{31} \quad (b_1, c_1, c_2 \geq 1;\ b_2, b_3, b_4, c_3, c_4, c_5, c_6 \geq 0)$,
(10) $\wedge$ **(13)** : $w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}$
$(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32} \quad (b_1, e_1, e_4 \geq 1;\ b_2, b_3, b_4, e_2e_3, e_5, e_6, e_7, e_8 \geq 0)$,

$\mathbf{(10)} \wedge \mathbf{(19)}: w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}\mu_{31}^{-1}$
$(b_1 \geq 1;\ b_2, b_3, b_4 \geq 0)$,
$\mathbf{(10)} \wedge \mathbf{(6)}: w = \mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4-1}$
$\mu_{12}\mu_{32}\mu_{31}\mu_{21} \quad (b_1, b_4 \geq 1;\ b_2, b_3 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(3)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}\mu_{32}\mu_{12} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(5)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}\mu_{21}\mu_{23} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(8)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21} \quad (c_1, c_2, x \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(9)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$(c_1, c_2, a_1 \geq 1;\ c_3, c_4, c_5, c_6, a_2, a_3, a_4 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(11)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1'}(\mu_{12}\mu_{32})^{c_2'}(\mu_{23}\mu_{21})^{c_3'}\cdots(\mu_{12}\mu_{21})^{c_4'}$
$(\mu_{12}\mu_{32})^{c_5'}(\mu_{23}\mu_{21})^{c_6'}\mu_{31} \quad (c_1, c_2, c_1', c_2' \geq 1;\ c_3, c_4, c_5, c_6, c_3', c_4', c_5', c_6' \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(13)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots$
$(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}$
$(c_1, c_2, e_1, e_4 \geq 1;\ c_3, c_4, c_5, c_6, e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
$\mathbf{(11)} \wedge \mathbf{(19)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$
$(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}\mu_{31}^{-1} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$.

In the relation **(11)**, if we take $c_6 = 0$ and $c_5 \geq 1$ then we have the intersection **(11)** $\wedge$ **(6)** as follows.

$$\mathbf{(11)} \wedge \mathbf{(6)}: w = \mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}$$
$$(\mu_{12}\mu_{32})^{c_5-1}\mu_{12}\mu_{32}\mu_{31}\mu_{21} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0).$$

Similarly, in the relation **(12)**, if $d_6 = 0$ and $d_5 \geq 1$ then we have

$$\mathbf{(12)}\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$$
$$(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}\mu_{31}\mu_{32} - \mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}$$
$$(\mu_{12}\mu_{13})^{d_3}\cdots(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{31}\mu_{21}$$
$$(d_1, d_2, d_5 \geq 1;\ d_3, d_4, d_6 \geq 0),$$

and we get the intersection composition **(12)** $\wedge$ **(3)** as follows.

$$\mathbf{(12)} \wedge \mathbf{(3)}: w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$$
$$(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}\mu_{31}\mu_{32}\mu_{12}.$$

We also have the following ambiguities.

(12) $\wedge$ **(4)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}\mu_{12}\mu_{13} \quad (d_1, d_2 \geq 1;\ d_3, d_4, d_5, d_6 \geq 0)$,
(12) $\wedge$ **(6)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}\mu_{31}\mu_{21} \quad (d_1, d_2 \geq 1;\ d_3, d_4, d_5, d_6 \geq 0)$,
(12) $\wedge$ **(10)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$(d_1, d_2, b_1 \geq 1;\ d_3, d_4, d_5, d_6, b_2, b_3, b_4 \geq 0)$,
(12) $\wedge$ **(12)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d'_1}(\mu_{21}\mu_{31})^{d'_2}(\mu_{13}\mu_{12})^{d'_3}\cdots(\mu_{21}\mu_{12})^{d'_4}$
$(\mu_{21}\mu_{31})^{d'_5}(\mu_{13}\mu_{12})^{d'_6}\mu_{32} \quad (d_1, d_2, d'_1, d'_2 \geq 1;\ d_3, d_4, d_5, d_6, d'_3, d'_4, d'_5, d'_6 \geq 0)$,
(12) $\wedge$ **(14)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}$
$(d_1, d_2, f_1, f_4 \geq 1;\ d_3, d_4, d_5, d_6, f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(12) $\wedge$ **(20)** : $w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$
$(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}\mu_{32}^{-1} \quad (d_1, d_2 \geq 1;\ d_3, d_4, d_5, d_6 \geq 0)$,
(13) $\wedge$ **(4)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}\mu_{12}\mu_{13}\ (e_1, e_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(13) $\wedge$ **(6)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}\mu_{31}\mu_{21} \quad (e_1, e_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(13) $\wedge$ **(10)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}$
$(\mu_{12}\mu_{32})^{b_4}\mu_{31} \quad (e_1, e_4, b_1 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8, b_2, b_3, b_4 \geq 0)$,
(13) $\wedge$ **(12)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots$
$(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}$
$(e_1, e_4, d_1, d_2 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8, d_3, d_4, d_5, d_6 \geq 0)$,
(13) $\wedge$ **(14)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}$
$(e_1, e_4, f_1, f_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8, f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(13) $\wedge$ **(20)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}$
$(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}\mu_{32}^{-1} \quad (e_1, e_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,

(13) $\wedge$ **(3)** $: w = \mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots$
$(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8-1}\mu_{21}\mu_{31}\mu_{32}\mu_{12}$
$(e_1, e_4, \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(14) $\wedge$ **(3)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}\mu_{32}\mu_{12}$
$(f_1, f_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(14) $\wedge$ **(5)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}\mu_{21}\mu_{23}$
$(f_1, f_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(14) $\wedge$ **(8)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21}$
$(f_1, f_4, x \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(14) $\wedge$ **(9)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots$
$(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$(f_1, f_4, a_1 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8, a_2, a_3, a_4 \geq 0)$,
(14) $\wedge$ **(11)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}$
$(\mu_{23}\mu_{21})^{c_3}\cdots(\mu_{12}\mu_{21})^{c_4}(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31}$
$(f_1, f_4, c_1, c_2 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8, c_3, c_4, c_5, c_6 \geq 0)$,
(14) $\wedge$ **(13)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}$
$(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}$
$(f_1, f_4, e_1, e_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8, e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(14) $\wedge$ **(19)** $: w = \mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}\mu_{31}^{-1}$
$(f_1, f_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(15) $\wedge$ **(21)** $: w = \mu_{12}\mu_{12}^{-1}\mu_{12}$, **(16)** $\wedge$ **(22)** $: w = \mu_{13}\mu_{13}^{-1}\mu_{13}$,
(18) $\wedge$ **(24)** $: w = \mu_{23}\mu_{23}^{-1}\mu_{23}$, **(19)** $\wedge$ **(25)** $: w = \mu_{31}\mu_{31}^{-1}\mu_{31}$,
(21) $\wedge$ **(15)** $: w = \mu_{12}^{-1}\mu_{12}\mu_{12}^{-1}$, **(22)** $\wedge$ **(16)** $: w = \mu_{13}^{-1}\mu_{13}\mu_{13}^{-1}$,
(23) $\wedge$ **(2)** $: w = \mu_{21}^{-1}\mu_{21}\mu_{23}\mu_{13}$, **(23)** $\wedge$ **(17)** $: w = \mu_{21}^{-1}\mu_{21}\mu_{21}^{-1}$,
(17) $\wedge$ **(23)** $: w = \mu_{21}\mu_{21}^{-1}\mu_{21}$, **(20)** $\wedge$ **(26)** $: w = \mu_{32}\mu_{32}^{-1}\mu_{32}$,
(23) $\wedge$ **(7)** $: w = \mu_{21}^{-1}\mu_{21}(\mu_{12}\mu_{13})^{x}\mu_{23}$ $(x \geq 1)$,
(24) $\wedge$ **(1)** $: w = \mu_{23}^{-1}\mu_{23}\mu_{13}\mu_{12}$, **(24)** $\wedge$ **(18)** $: w = \mu_{23}^{-1}\mu_{23}\mu_{23}^{-1}$,
(25) $\wedge$ **(3)** $: w = \mu_{31}^{-1}\mu_{31}\mu_{32}\mu_{12}$, **(25)** $\wedge$ **(5)** $: w = \mu_{31}^{-1}\mu_{31}\mu_{21}\mu_{23}$,
(25) $\wedge$ **(8)** $: w = \mu_{31}^{-1}\mu_{31}(\mu_{13}\mu_{23})^{x}\mu_{21}$ $(x \geq 1)$,

(25) $\wedge$ **(9)** : $w = \mu_{31}^{-1}\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}$
$(\mu_{21}\mu_{31})^{a_4}\mu_{32} \quad (a_1 \geq 1;\ a_2, a_3, a_4 \geq 0)$,
(25) $\wedge$ **(11)** : $w = \mu_{31}^{-1}\mu_{31}\mu_{13}(\mu_{12}\mu_{21})^{c_1}(\mu_{12}\mu_{32})^{c_2}(\mu_{23}\mu_{21})^{c_3}\cdots$
$(\mu_{12}\mu_{21})^{c_4}(\mu_{12}\mu_{32})^{c_5}(\mu_{23}\mu_{21})^{c_6}\mu_{31} \quad (c_1, c_2 \geq 1;\ c_3, c_4, c_5, c_6 \geq 0)$,
(25) $\wedge$ **(13)** : $w = \mu_{31}^{-1}\mu_{31}(\mu_{13}\mu_{12})^{e_1}(\mu_{21}\mu_{12})^{e_2}(\mu_{32}\mu_{12})^{e_3}(\mu_{21}\mu_{31})^{e_4}\cdots$
$(\mu_{13}\mu_{12})^{e_5}(\mu_{21}\mu_{12})^{e_6}(\mu_{32}\mu_{12})^{e_7}(\mu_{21}\mu_{31})^{e_8}\mu_{32}$
$(e_1, e_4 \geq 1;\ e_2, e_3, e_5, e_6, e_7, e_8 \geq 0)$,
(25) $\wedge$ **(19)** : $w = \mu_{31}^{-1}\mu_{31}\mu_{31}^{-1}$, **(26)** $\wedge$ **(4)** : $w = \mu_{32}^{-1}\mu_{32}\mu_{12}\mu_{13}$,
(26) $\wedge$ **(6)** : $w = \mu_{32}^{-1}\mu_{32}\mu_{31}\mu_{21}$,
(26) $\wedge$ **(10)** : $w = \mu_{32}^{-1}\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}$
$(\mu_{12}\mu_{32})^{b_4}\mu_{31} \quad (b_1 \geq 1;\ b_2, b_3, b_4 \geq 0)$,
(26) $\wedge$ **(12)** : $w = \mu_{32}^{-1}\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots$
$(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5}(\mu_{13}\mu_{12})^{d_6}\mu_{32}$
$(d_1, d_2 \geq 1;\ d_3, d_4, d_5, d_6 \geq 0)$,
(26) $\wedge$ **(14)** : $w = \mu_{32}^{-1}\mu_{32}(\mu_{23}\mu_{21})^{f_1}(\mu_{12}\mu_{21})^{f_2}(\mu_{31}\mu_{21})^{f_3}(\mu_{12}\mu_{32})^{f_4}\cdots$
$(\mu_{23}\mu_{21})^{f_5}(\mu_{12}\mu_{21})^{f_6}(\mu_{31}\mu_{21})^{f_7}(\mu_{12}\mu_{32})^{f_8}\mu_{31}$
$(f_1, f_4 \geq 1;\ f_2, f_3, f_5, f_6, f_7, f_8 \geq 0)$,
(26) $\wedge$ **(20)** : $w = \mu_{32}^{-1}\mu_{32}\mu_{32}^{-1}$.

All these ambiguities are trivial. Let us show some of them as follows.

(9) $\wedge$ **(10)** : $w = \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}$
$\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$(a_1, b_1 \geq 1;\ a_2, a_3, a_4, b_2, b_3, b_4 \geq 0)$,
$(f,g)_w = (\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{32}$
$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4})$
$(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31} -$
$\mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}$
$(\mu_{32}(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}(\mu_{21}\mu_{23})^{b_1-1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4})$
$= \mu_{31}(\mu_{13}\mu_{12})^{a_1}(\mu_{21}\mu_{31})^{a_2}\cdots(\mu_{13}\mu_{12})^{a_3}(\mu_{21}\mu_{31})^{a_4}\mu_{21}\mu_{31}\mu_{32}\mu_{23}$
$(\mu_{21}\mu_{23})^{b_1-1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$
$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$
$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$
$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$
$\mu_{31}\mu_{21}\mu_{23}(\mu_{21}\mu_{23})^{b_1-1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$
$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$
$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$

$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$\mu_{23}\mu_{21}\mu_{31}(\mu_{21}\mu_{23})^{b_1-1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$$
$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{2}$$
$$\mu_{31}\mu_{23}\mu_{21}(\mu_{21}\mu_{23})^{b_1-2}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}.$$

If the reduction process is repeated b_1 times then we get

$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}\mu_{31}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$$
$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}\mu_{31}\mu_{32}\mu_{12}(\mu_{32}\mu_{12})^{b_2-1}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$$
$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})\mu_{31}(\mu_{32}\mu_{12})^{b_2-1}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}$$
$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{2}\mu_{31}(\mu_{32}\mu_{12})^{b_2-2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}.$$

Similarly, we repeat reduction processes b_2 times then we obtain

$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}(\mu_{32}\mu_{12})^{b_2}\cdots\mu_{31}(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}.$$

We obtain the relation as follows when we continous reduction processes b_3 times.

$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}\mu_{31}(\mu_{32}\mu_{12})^{b_4}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}(\mu_{23}\mu_{21})^{b_1}$$
$$(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31}.$$

Lastly, we repeat the same process b_4 times then we get

$$\equiv \mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}(\mu_{32}\mu_{12})^{b_2}\cdots(\mu_{21}\mu_{23})^{b_3}(\mu_{32}\mu_{12})^{b_4}\mu_{31}$$
$$-\mu_{12}\mu_{32}\mu_{31}\mu_{13}(\mu_{12}\mu_{13})^{a_1-1}(\mu_{31}\mu_{21})^{a_2}\cdots(\mu_{12}\mu_{13})^{a_3}(\mu_{31}\mu_{21})^{a_4}$$
$$(\mu_{23}\mu_{21})^{b_1}(\mu_{12}\mu_{32})^{b_2}\cdots(\mu_{23}\mu_{21})^{b_3}(\mu_{12}\mu_{32})^{b_4}\mu_{31} \equiv 0.$$

Now we give more examples as follows.

$$\begin{aligned}
\mathbf{(23)} \wedge \mathbf{(2)} : w &= \mu_{21}^{-1}\mu_{21}\mu_{23}\mu_{13} \\
(f,g)_w &= (\mu_{21}^{-1}\mu_{21}-1)\mu_{23}\mu_{13} - \mu_{21}^{-1}(\mu_{21}\mu_{23}\mu_{13} - \mu_{13}\mu_{23}\mu_{21}) \\
&= \mu_{21}^{-1}\mu_{13}\mu_{23}\mu_{21} - \mu_{23}\mu_{13} \\
&\equiv \mu_{21}\mu_{21}^{-1}\mu_{13}\mu_{23}\mu_{21} - \mu_{21}\mu_{23}\mu_{13} \\
&\equiv \mu_{13}\mu_{23}\mu_{21} - \mu_{21}\mu_{23}\mu_{13} \equiv 0.
\end{aligned}$$

$$\mathbf{(12)} \wedge \mathbf{(3)} : w = \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots$$
$$(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}\mu_{31}\mu_{32}\mu_{12}$$
$$(f,g)_w = (\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots$$
$$(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}\mu_{31}\mu_{32}$$
$$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{31}\mu_{21})\mu_{12}$$
$$-\mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots$$
$$(\mu_{21}\mu_{12})^{d_4}(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}(\mu_{31}\mu_{32}\mu_{12} - \mu_{12}\mu_{32}\mu_{31})$$
$$= \mu_{32}\mu_{23}(\mu_{21}\mu_{12})^{d_1}(\mu_{21}\mu_{31})^{d_2}(\mu_{13}\mu_{12})^{d_3}\cdots(\mu_{21}\mu_{12})^{d_4}$$
$$(\mu_{21}\mu_{31})^{d_5-1}\mu_{21}\mu_{12}\mu_{32}\mu_{31}$$
$$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{31}\mu_{21}\mu_{12}$$
$$\equiv \mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{12}\mu_{21}\mu_{31}$$
$$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{31}\mu_{21}\mu_{12}$$
$$\equiv \mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{12}\mu_{21}\mu_{31}$$
$$-\mu_{21}\mu_{31}\mu_{32}\mu_{23}\mu_{32}(\mu_{12}\mu_{21})^{d_1}(\mu_{31}\mu_{21})^{d_2-1}(\mu_{12}\mu_{13})^{d_3}\cdots$$
$$(\mu_{12}\mu_{21})^{d_4}(\mu_{31}\mu_{21})^{d_5-1}\mu_{12}\mu_{21}\mu_{31} \equiv 0.$$

It is seen that there are no any inclusion compositions of relations $(1)-(26)$. Hence the result. □

Now we present second main result of this section. In the following result, we give a Gröbner-Shirshov basis of pure virtual braid group PV_n for $n \geq 4$. Since

$n \geq 4$ we use the relation $\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij} \quad (\{i,j\} \cap \{k,l\} = \emptyset)$ given in (2) to obtain Gröbner-Shirshov basis. As seen this type of relation makes the Gröbner-Shirshov basis quite complicated.

THEOREM 2.2. *A Gröbner-Shirshov basis of pure virtual braid group PV_n for $n \geq 4$ consists of the following relations:*

For $i > k > j$ *and* $i \geq z_3 > z_2 > z_1 \geq j$

$\{a_1, a_2\} \cap \{k, i\} = \emptyset$ *and* $\{b_1, b_2\} \cap \{k, j\} = \emptyset$ *and*

$\{c_1, c_2\} \cap \{i, k\} = \emptyset$

(1_A) $\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y,$

(1_B) $\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kj} - \mu_{ki}\mu_{kj}\mu_{ij}\mu_{ji}(\mu_{z_2z_3}\mu_{z_1z_3})^x$ $(\mu_{b_1b_2})^y,$

(1_C) $\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik} - \mu_{jk}\mu_{ik}\mu_{ij}\mu_{ji}(\mu_{z_3z_1}\mu_{z_2z_1})^x$ $(\mu_{c_1c_2})^y,$

For $k > i > j$ *and* $k \geq z_3 > z_2 > z_1 \geq j$

$\{a_1, a_2\} \cap \{i, k\} = \emptyset$ *and* $\{b_1, b_2\} \cap \{i, j\} = \emptyset$

(2_A) $\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik} - \mu_{jk}\mu_{ik}\mu_{ij}\mu_{ji}(\mu_{z_1z_3}\mu_{z_1z_2})^x$ $(\mu_{a_1a_2})^y,$

(2_B) $\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_3z_2}\mu_{z_3z_1})^x(\mu_{b_1b_2})^y,$

For $i > j > k$ *and* $i \geq z_3 > z_2 > z_1 \geq k$ *and*

$\{a_1, a_2\} \cap \{i, k\} = \emptyset$

(3) $\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik} - \mu_{jk}\mu_{ik}\mu_{ij}\mu_{ji}(\mu_{z_2z_1}\mu_{z_2z_3})^x(\mu_{a_1a_2})^y,$

For $l > i > k > j$ *and* $l \geq z_3 > z_2 > z_1 \geq j$

$\{a_1, a_2\} \cap \{k, i\} = \emptyset$ *and* $\{b_1, b_2\} \cap \{k, l\} = \emptyset$

(4_A) $\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}\mu_{jl}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y,$

(4_B) $\mu_{ij}\mu_{jk}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{kl} - \mu_{kl}\mu_{ij}\mu_{jl}\mu_{jk}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{b_1b_2})^y,$

For $l > k > i > j$ *and* $l \geq z_3 > z_2 > z_1 \geq j$ *and*

$\{a_1, a_2\} \cap \{i, j\} = \emptyset$

(5) $\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y,$

For $j > i > k > l$ *and* $j \geq z_3 > z_2 > z_1 \geq l$

$\{a_1, a_2\} \cap \{k, i\} = \emptyset$ *and* $\{b_1, b_2\} \cap \{k, l\} = \emptyset$

(6_A) $\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y,$

(6_B) $\mu_{ij}\mu_{kj}\mu_{lj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}\mu_{lj}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y,$

(6_C) $\mu_{ij}\mu_{li}\mu_{ki}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl} - \mu_{kl}\mu_{ij}\mu_{ki}\mu_{li}(\mu_{z_3z_2}\mu_{z_1z_2})^x(\mu_{b_1b_2})^y$,

(6_D) $\mu_{ij}\mu_{lj}\mu_{kj}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl} - \mu_{kl}\mu_{ij}\mu_{kj}\mu_{lj}(\mu_{z_3z_2}\mu_{z_1z_2})^x(\mu_{b_1b_2})^y$,

$\overline{\underline{\text{For } j > k > i > l}}$ *and* $\overline{\underline{j \geq z_3 > z_2 > z_1 \geq l}}$ *and*

$\overline{\underline{\{a_1, a_2\} \cap \{i, j\} = \emptyset}}$

(7_A) $\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y$,

(7_B) $\mu_{ki}\mu_{kj}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}\mu_{lk}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y$,

$\overline{\underline{\text{For } i > j > k > l}}$ *and* $\overline{\underline{i \geq z_3 > z_2 > z_1 \geq l}}$ *and*

$\overline{\underline{i \geq p_2 > p_1 \geq l}}$

$\overline{\underline{\{a_1, a_2\} \cap \{k, i\} = \emptyset}}$ *and* $\overline{\underline{\{b_1, b_2\} \cap \{k, l\} = \emptyset}}$

(8_A) $\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y$,

(8_B) $\mu_{ij}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki} -$
$\mu_{ki}\mu_{kj}\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{a_1a_2})^y$,

(8_C) $\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_1p_2})^{x_1}\mu_{ki}(\mu_{a_1a_2})^y -$
$\mu_{ki}\mu_{kj}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{p_1p_2})^{x_1}(\mu_{a_1a_2})^y$,

(8_D) $\mu_{ij}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl} - \mu_{kl}\mu_{ij}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y$,

$\overline{\underline{\text{For } i > k > l > j}}$ *and* $\overline{\underline{i \geq z_3 > z_2 > z_1 \geq j}}$ *and*

$\overline{\underline{i \geq p_4 > p_3 > p_2 > p_1 \geq j}}$ *and* $\overline{\underline{i \geq u_3 > u_2 > u_1 \geq j}}$ *and*

$\overline{\underline{\{a_1, a_2\} \cap \{k, i\} = \emptyset}}$ *and* $\overline{\underline{\{b_1, b_2\} \cap \{k, l\} = \emptyset}}$

(9_A) $\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki} -$
$\mu_{ki}\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_3}\mu_{z_1z_2})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y$,

(9_B) $\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl} -$
$\mu_{kl}\mu_{ij}(\mu_{u_3u_1}\mu_{u_2u_1})^{x_1}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y$,

(9_C) $\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl} -$
$\mu_{kl}\mu_{ij}(\mu_{u_1u_2}\mu_{u_1u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y$,

(9_D) $\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^x(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^y\mu_{kl} -$
$\mu_{kl}\mu_{ij}\mu_{ji}(\mu_{u_2u_3}\mu_{u_1u_3})^x(\mu_{z_3z_1}\mu_{z_2z_1})^{x_1}(\mu_{b_1b_2})^y$,

$\overline{\underline{\text{For } k > l > i > j}}$ *and* $\overline{\underline{k \geq z_3 > z_2 > z_1 \geq i}}$ *and*

$\overline{\underline{\{a_1, a_2\} \cap \{i, j\} = \emptyset}}$

(10_A) $\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{a_1a_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y$,

(10_B) $\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{a_1a_2})^y\mu_{ij} -$
$\mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}(\mu_{z_3z_1}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y$,

$\overline{\underline{\text{For } k > j > i > l}}$ *and* $\overline{\underline{k \geq z_3 > z_2 > z_1 \geq l}}$

$\overline{\underline{k \geq p_4 > p_3 > p_2 > p_1 \geq l}}$ *and* $\overline{\underline{\{a_1, a_2\} \cap \{i, j\} = \emptyset}}$

$$(11)\quad \mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_3}\mu_{z_1z_2})^{x}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y},$$

$\underline{\text{For } k > i > l > j}$ *and* $\underline{k \geq z_3 > z_2 > z_1 \geq j}$ *and* $\underline{k \geq p_4 > p_3 > p_2 > p_1 \geq j}$ *and* $\underline{k \geq u_3 > u_2 > u_1 \geq j}$ *and* $\underline{\{a_1, a_2\} \cap \{i, j\} = \emptyset}$

$$(12_A)\quad \mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_3u_2}\mu_{u_1u_2})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y},$$

$$(12_B)\quad \mu_{ki}\mu_{kj}(\mu_{u_1u_3}\mu_{u_2u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_2u_3}\mu_{u_1u_3})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y},$$

$$(12_C)\quad \mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{p_2p_1})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y},$$

$\underline{\text{For } i > k}$ *and* $\underline{\{i, j\} \cap \{k, l\} = \emptyset}$

$$(13)\quad \mu_{ij}\mu_{kl} - \mu_{kl}\mu_{ij},$$

$\underline{\text{For } 1 \leq r \neq s \leq n}$

$$(14_A)\quad \mu_{rs}\mu_{rs}^{-1} - 1, \qquad (14_B)\mu_{rs}^{-1}\mu_{rs} - 1,$$

where $x, x_1, x_2, y \geq 0$.

Proof. We need to prove that all compositions among relations $(1_A) - (14_B)$ are trivial. To do that, firstly, we consider the intersection compositions of these relations. It should be noted that the compositions to be written below are the compositions that occur for $n = 4$. Certainly, the relations in Theorem 2.2 are written for $n \geq 4$, and are provided for $n \geq 4$. Here we would like to be more understandable and applicable. Considering $n \geq 4$, it comes with some similar additional intersections and they are also trivial.

For $x > 1$ in any relation above, let us take $x = 2$ in the relation (1_A) then we have

$$(1_A)\quad \mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})(\mu_{z_1z_2}\mu_{z_1z_3})(\mu_{a_1a_2})^{y}\mu_{ki} - \mu_{ki}\mu_{kj}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})(\mu_{z_1z_3}\mu_{z_1z_2})(\mu_{a_1a_2})^{y}.$$

Note that the words $(\mu_{z_1z_2}\mu_{z_1z_3})$ and $(\mu_{z_1z_2}\mu_{z_1z_3})$ are just at the same word form. These words can be different words according to corresponding order given in (1_A). The logic described here is valid for all x, x_1, x_2 and y in this theorem. Hence, we have the following ambiguties w:

$$(1_A) \wedge (6_A) \quad : \quad w = \mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{ti}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},$$

$$(1_A) \wedge (6_B) \quad : \quad w = \mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{ti}\mu_{li}$$

$$
\begin{aligned}
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},\\
(1_A)\wedge(6_C) &: && w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{tk}\mu_{pk}\\
& && (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^{y}\mu_{pt},\\
(1_A)\wedge(6_D) &: && w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{ti}\mu_{pi}\\
& && (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^{y}\mu_{pt},\\
(1_A)\wedge(13) &: && w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{tl},\\
(1_A)\wedge(14_A) &: && w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ki}\mu_{ki}^{-1},\\
(1_B)\wedge(2_A) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{jk}\mu_{jt}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(1_B)\wedge(2_B) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{kt}\\
& && (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y'}\mu_{jt},\\
(1_B)\wedge(3) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{jk}\mu_{jt}\\
& && (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(1_B)\wedge(13) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{tl},\\
(1_B)\wedge(14_A) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{kj}^{-1},\\
(1_B)\wedge(5) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{kt}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{jt},\\
(1_B)\wedge(7_A) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{kt}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{jt},\\
(1_B)\wedge(7_B) &: && w=\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kj}\mu_{kt}\mu_{lk}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{jt},\\
(1_C)\wedge(2_A) &: && w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{c_1c_2})^{y}\mu_{ik}\mu_{ki}\mu_{kt}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{it},\\
(1_C)\wedge(2_B) &: && w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{c_1c_2})^{y}\mu_{ik}\mu_{it}\\
& && (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y}\mu_{kt},\\
(1_C)\wedge(3) &: && w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{c_1c_2})^{y}\mu_{ik}\mu_{ki}\mu_{kt}\\
& && (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^{y}\mu_{it},\\
(1_C)\wedge(5) &: && w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{c_1c_2})^{y}\mu_{ik}\mu_{it}\\
& && (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(1_C)\wedge(7_A) &: && w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{c_1c_2})^{y}\mu_{ik}\mu_{it}
\end{aligned}
$$

$$(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{kt},$$

$$(1_C)\wedge(7_B) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{it}\mu_{li}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{kt},$$

$$(1_C)\wedge(8_A) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{tk}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{ti},$$

$$(1_C)\wedge(8_B) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{tk}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{ti},$$

$$(1_C)\wedge(8_C) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{tk}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{p_1p_2})^{x_1}(\mu_{a_1a_2})^y\mu_{ti},$$

$$(1_C)\wedge(8_D) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}(\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^y\mu_{tl},$$

$$(1_C)\wedge(12_A) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{it}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\mu_{kt},$$

$$(1_C)\wedge(12_B) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{it}(\mu_{u_1u_3}\mu_{u_2u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\mu_{kt},$$

$$(1_C)\wedge(12_C) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{it}(\mu_{p_1p_2})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{kt},$$

$$(1_C)\wedge(13) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{tl},$$

$$(1_C)\wedge(14_A) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik}\mu_{ik}^{-1},$$

$$(2_A)\wedge(6_A) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tk}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{ti},$$

$$(2_A)\wedge(6_B) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tk}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{ti},$$

$$(2_A)\wedge(6_C) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{li}\mu_{ti}(\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},$$

$$(2_A)\wedge(6_D) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{lk}\mu_{tk}(\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},$$

$$(2_A)\wedge(13) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tl},$$

$$(2_A)\wedge(14_A) \quad : \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{ik}^{-1},$$

$$(2_B)\wedge(2_A) \quad : \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{ik},$$

$$
\begin{aligned}
(2_B)\wedge(2_B) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{it} \\
&\qquad (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y'}\mu_{jt}, \\
(2_B)\wedge(3) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{ji}\mu_{jk} \\
&\qquad (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^y\mu_{ik}, \\
(2_B)\wedge(5) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{it} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}, \\
(2_B)\wedge(7_A) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{it} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}, \\
(2_B)\wedge(7_B) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{it}\mu_{li} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}, \\
(2_B)\wedge(13) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{tl}, \\
(2_B)\wedge(14_A) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{ij}^{-1}, \\
(3)\wedge(1_A) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tk} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{ti}, \\
(3)\wedge(1_B) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik} \\
&\qquad \mu_{ki}\mu_{ti}\mu_{ki}(\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^y\mu_{tk}, \\
(3)\wedge(1_C) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{ki}\mu_{kt} \\
&\qquad (\mu_{z_2z_1}\mu_{z_3z_1})^{x'}(\mu_{c_1c_2})^y\mu_{it}, \\
(3)\wedge(2_A) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{ki}\mu_{kt} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{it}, \\
(3)\wedge(2_B) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{it} \\
&\qquad (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^y\mu_{kt}, \\
(3)\wedge(4_A) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tk}\mu_{kl} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{ti}, \\
(3)\wedge(4_B) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{kt}\mu_{kl} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{b_1b_2})^y\mu_{tl}, \\
(3)\wedge(9_A) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}(\mu_{p_1p_2})^{x_1} \\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y'}\mu_{ti}, \\
(3)\wedge(9_B) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik} \\
&\qquad (\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^y\mu_{tl}, \\
(3)\wedge(9_C) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik} \\
&\qquad (\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^y\mu_{tl},
\end{aligned}
$$

$$
\begin{aligned}
(3)\wedge(9_D) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{ki}\\
&\qquad (\mu_{u_1u_3}\mu_{u_2u_3})^{x'}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^y\mu_{tl},\\
(3)\wedge(11) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{it}\\
&\qquad (\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^{y'}\mu_{kt},\\
(3)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{tl},\\
(3)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik}\mu_{ik}^{-1},\\
(4_A)\wedge(6_A) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ti}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},\\
(4_A)\wedge(6_B) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ti}\mu_{li}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},\\
(4_A)\wedge(6_C) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{lk}\mu_{tk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},\\
(4_A)\wedge(6_D) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{li}\mu_{ti}\\
&\qquad (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},\\
(4_A)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(4_A)\wedge(14) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(4_B)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{jk}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{tp},\\
(4_B)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{jk}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kl}^{-1},\\
(5)\wedge(13) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(5)\wedge(14_A) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(6_A)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(6_A)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(6_B)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{lj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(6_B)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{kj}\mu_{lj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(6_C)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{li}\mu_{ki}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{tp},\\
(6_C)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{li}\mu_{ki}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kl}^{-1},\\
(6_D)\wedge(13) \quad &: \quad w=\mu_{ij}\mu_{lj}\mu_{kj}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{tp},\\
(6_D)\wedge(14_A) \quad &: \quad w=\mu_{ij}\mu_{lj}\mu_{kj}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kl}^{-1},\\
(7_A)\wedge(13) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(7_A)\wedge(14_A) \quad &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(7_B)\wedge(13) \quad &: \quad w=\mu_{ki}\mu_{kj}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(7_B)\wedge(14_A) \quad &: \quad w=\mu_{ki}\mu_{kj}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},
\end{aligned}
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\begin{aligned}
(8_A)\wedge(13) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(8_A)\wedge(14_A) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(8_B)\wedge(13) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(8_B)\wedge(14_A) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(8_C)\wedge(13) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_1p_2})^{x_1}(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(8_C)\wedge(14_A) &: \quad w=\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_1p_2})^{x_1}(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(8_D)\wedge(2_A) &: \quad w=\mu_{ij}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{kt},\\
(8_D)\wedge(13) &: \quad w=\mu_{ij}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{tp},\\
(8_D)\wedge(14_A) &: \quad w=\mu_{ij}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kl}^{-1},\\
(9_A)\wedge(6_A) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{ti}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},\\
(9_A)\wedge(6_B) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{ti}\mu_{li}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{tk},\\
(9_A)\wedge(6_C) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{lk}\mu_{tk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},\\
(9_A)\wedge(6_D) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{li}\mu_{ti}\\
&\qquad (\mu_{z_1z_2}\mu_{z_3z_2})^{x'}(\mu_{b_1b_2})^y\mu_{tl},\\
(9_A)\wedge(13) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{tp},\\
(9_A)\wedge(14_A) &: \quad w=\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki}\mu_{ki}^{-1},\\
(9_B)\wedge(2_A) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{kt},\\
(9_B)\wedge(2_B) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y'}\mu_{lt},\\
(9_B)\wedge(3) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^y\mu_{kt},\\
(9_B)\wedge(5) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{lt},\\
(9_B)\wedge(7_A) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{lt},\\
(9_B)\wedge(7_B) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl}\mu_{kt}\mu_{nk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{lt},
\end{aligned}
$$

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\begin{aligned}
(9_B)\wedge(13) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{tp},\\
(9_B)\wedge(14_A) &: \quad w=\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kl}^{-1},\\
(9_C)\wedge(2_A) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(9_C)\wedge(2_B) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y'}\mu_{lt},\\
(9_C)\wedge(3) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(9_C)\wedge(5) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_C)\wedge(7_A) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_C)\wedge(7_B) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\mu_{nk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_C)\wedge(13) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{tp},\\
(9_C)\wedge(14_A) &: \quad w=\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kl}^{-1},\\
(9_D)\wedge(2_A) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(9_D)\wedge(2_B) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_3}\mu_{z_2z_3})^{x'}(\mu_{b_1b_2})^{y'}\mu_{lt},\\
(9_D)\wedge(3) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{lk}\mu_{lt}\\
&\qquad (\mu_{z_2z_3}\mu_{z_2z_1})^{x'}(\mu_{a_1a_2})^{y}\mu_{kt},\\
(9_D)\wedge(5) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_D)\wedge(7_A) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_D)\wedge(7_B) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kt}\mu_{nk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y}\mu_{lt},\\
(9_D)\wedge(13) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{tp},\\
(9_D)\wedge(14_A) &: \quad w=\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^{x}(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^{y}\mu_{kl}\mu_{kl}^{-1},\\
(10_A)\wedge(2_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^{x}(\mu_{a_1a_2})^{y}\mu_{ij}\mu_{jk}\\
&\qquad (\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^{y'}\mu_{ik},
\end{aligned}
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\begin{aligned}
(10_A)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(10_A)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(10_B)\wedge(2_A) &: \quad w=\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ji}\mu_{jk}\\
& \quad\ (\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^{y'}\mu_{ik},\\
(10_B)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(10_B)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(11)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(11)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(12_A)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\\
& \quad\ \mu_{ij}\mu_{tp},\\
(12_A)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\\
& \quad\ \mu_{ij}\mu_{ij}^{-1},\\
(12_B)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{u_1u_3}\mu_{u_2u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\\
& \quad\ \mu_{ij}\mu_{tp},\\
(12_B)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{u_1u_3}\mu_{u_2u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\\
& \quad\ \mu_{ij}\mu_{ij}^{-1},\\
(12_C)\wedge(13) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij}\mu_{tp},\\
(12_C)\wedge(14_A) &: \quad w=\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij}\mu_{ij}^{-1},\\
(13)\wedge(2_A) &: \quad w=\mu_{kt}\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik},\\
(13)\wedge(4_A) &: \quad w=\mu_{lt}\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(13)\wedge(4_B) &: \quad w=\mu_{lt}\mu_{ij}\mu_{jk}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(13)\wedge(5) &: \quad w=\mu_{lt}\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(13)\wedge(7_A) &: \quad w=\mu_{jt}\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(13)\wedge(7_B) &: \quad w=\mu_{jt}\mu_{ki}\mu_{kj}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(13)\wedge(13) &: \quad w=\mu_{ij}\mu_{kl}\mu_{tp},\\
(13)\wedge(14_A) &: \quad w=\mu_{ij}\mu_{kl}\mu_{kl}^{-1},\\
(14_A)\wedge(14_B) &: \quad w=\mu_{rs}^{-1}\mu_{rs}\mu_{rs}^{-1},\\
(14_B)\wedge(1_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(1_B) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{ji}\mu_{ki}\mu_{ji}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kj},\\
(14_B)\wedge(1_C) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{c_1c_2})^y\mu_{ik},\\
(14_B)\wedge(2_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ik},\\
(14_B)\wedge(2_B) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij},\\
(14_B)\wedge(3) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{ji}\mu_{jk}(\mu_{z_2z_3}\mu_{z_2z_1})^x(\mu_{a_1a_2})^y\mu_{ik},\\
(14_B)\wedge(4_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(4_B) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{jk}\mu_{jl}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{kl},
\end{aligned}
$$

$$
\begin{aligned}
(14_B)\wedge(5) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(6_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(6_B) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}\mu_{lj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(6_C) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{li}\mu_{ki}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(6_D) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{lj}\mu_{kj}(\mu_{z_1z_2}\mu_{z_3z_2})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(7_A) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(7_B) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}\mu_{lk}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(8_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(8_B) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(8_C) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kj}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_1p_2})^{x_1}(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(8_D) &: \quad w=\mu_{ij}^{-1}\mu_{ij}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(9_A) &: \quad w=\mu_{ij}^{-1}\mu_{ij}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ki},\\
(14_B)\wedge(9_B) &: \quad w=\mu_{ij}^{-1}\mu_{ij}(\mu_{u_2u_1}\mu_{u_3u_1})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(9_C) &: \quad w=\mu_{ij}^{-1}\mu_{ij}(\mu_{u_1u_3}\mu_{u_1u_2})^{x_1}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(9_D) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{ji}(\mu_{u_1u_3}\mu_{u_2u_3})^x(\mu_{z_2z_1}\mu_{z_3z_1})^{x_1}(\mu_{b_1b_2})^y\mu_{kl},\\
(14_B)\wedge(10_A) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(10_B) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^x(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(11) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{z_1z_2}\mu_{z_1z_3})^x(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(12_A) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x\\
&\quad\quad (\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(12_B) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{u_1u_3}\mu_{u_2u_3})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^x\\
&\quad\quad (\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(12_C) &: \quad w=\mu_{ki}^{-1}\mu_{ki}\mu_{kj}(\mu_{p_1p_2})^{x_1}(\mu_{p_4p_3})^{x_2}(\mu_{a_1a_2})^y\mu_{ij},\\
(14_B)\wedge(13) &: \quad w=\mu_{ij}^{-1}\mu_{ij}\mu_{kl},\\
(14_B)\wedge(14_A) &: \quad w=\mu_{rs}^{-1}\mu_{rs}\mu_{rs}^{-1}.
\end{aligned}
$$

All these above ambiguties are trivial. Since to show these trivialities has taken so much times we have proven some of them below. The remaining parts can be shown in similar ways. Here, it is important to order notations given two relations. Firstly, we consider the ambigutie of relations (2_B) and (7_A). It should be noted here that the ordering $j > k > i > l$ is now $t > i > j > l$. The notational ordering for this ambigutie is as follows.

- Ordering for the relation (2_B): $k > i > j$ and $k \geq z_3 > z_2 > z_1 \geq j$ and $\{b_1, b_2\} \cap \{i, j\} = \emptyset$.

- Ordering for the relation (7_A): $t > i > j > l$ and $t \geq z_3 > z_2 > z_1 \geq l$ and $\{a_1, a_2\} \cap \{j, t\} = \emptyset$.

Thus we have

$$\begin{aligned}
&(2_B) \wedge (7_A) : w = \mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij}\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt},\\
&(f,g)_w = [\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{ij} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\\
&(\mu_{b_1b_2})^y]\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}\\
&-\mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y[\mu_{ij}\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt} -\\
&\mu_{jt}\mu_{it}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^{x'}(\mu_{a_1a_2})^y]\\
&= \mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\mu_{jt}\mu_{it}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^{x'}(\mu_{a_1a_2})^y\\
&-\mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}\\
&= \mu_{ki}\mu_{kj}(\mu_{z_1z_3}\mu_{z_2z_3})^x(\mu_{b_1b_2})^y\underbrace{\mu_{jt}\mu_{it}}_{*}\mu_{ij}(\mu_{z_1z_3}\mu_{z_1z_2})^{x'}(\mu_{a_1a_2})^y\\
&-\mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x(\mu_{b_1b_2})^y\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}(\mu_{a_1a_2})^y\mu_{jt}.
\end{aligned}$$

Let us take $y = 0$ to see the proof much easier. Since the word $*$ is of type $\mu_{z_1z_3}\mu_{z_2z_3}$, by using the relation (2_A) we can write,

$$\begin{aligned}
\equiv \quad & \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}\mu_{jt}(\mu_{z_1z_3}\mu_{z_1z_2})^{x'} -\\
& \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}\mu_{jt}.
\end{aligned}$$

In case of $x' = 0$, we have

$$\begin{aligned}
= \quad & \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}\mu_{jt} - \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}\mu_{jt}\\
\equiv \quad & 0.
\end{aligned}$$

In case of $x' \neq 0$, by using the order given for (7_A) we get

$$\begin{aligned}
\equiv \quad & \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}\mu_{jt} -\\
& \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{z_2z_3}\mu_{z_1z_3})^x\mu_{it}(\mu_{z_1z_2}\mu_{z_1z_3})^{x'}\mu_{jt} \equiv 0.
\end{aligned}$$

Let us consider the other one ambigutie. Ordering of this ambigutie;

- For the relation (12_A): $k > i > l > j$, $k \geq z_3 > z_2 > z_1 \geq j$, $k \geq p_4 > p_3 > p_2 > p_1 \geq j$, $k \geq u_3 > u_2 > u_1 \geq j$ and $\{a_1, a_2\} \cap \{i, j\} = \emptyset$.

- For the relation (13): $i > t$ and $\{t, p\} \cap \{i, j\} = \emptyset$.

$$
\begin{aligned}
&(12_A) \wedge (13) : w = \mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij}\mu_{tp},\\
&(f,g)_w = [\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{ij}\\
&-\mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_3u_2}\mu_{u_1u_2})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}]\mu_{tp}\\
&-\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}[\mu_{ij}\mu_{tp}-\mu_{tp}\mu_{ij}]\\
&= \underbrace{\mu_{ki}\mu_{kj}(\mu_{u_1u_2}\mu_{u_3u_2})^{x_1}(\mu_{z_2z_3}\mu_{z_2z_1})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{tp}\mu_{ij}}_{*}\\
&-\mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_3u_2}\mu_{u_1u_2})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{tp}.
\end{aligned}
$$

On account of the ordering given for the relation (13) and μ_{tp} is of the form $(\mu_{a_1a_2})$, the relation $*$ is of the form (12_A). Then we get

$$
\begin{aligned}
\equiv\ & \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_3u_2}\mu_{u_1u_2})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{tp}\\
-\ & \mu_{ij}\mu_{kj}\mu_{ki}(\mu_{u_3u_2}\mu_{u_1u_2})^{x_1}(\mu_{z_2z_1}\mu_{z_2z_3})^{x}(\mu_{p_2p_1})^{x_2}(\mu_{a_1a_2})^{y}\mu_{tp}\\
\equiv\ & 0.
\end{aligned}
$$

Now we consider the intersection composition of the relation (14_B) with (10_B). For $k > l > i > j$, $k \geq z_3 > z_2 > z_1 \geq i$ and $\{a_1, a_2\} \cap \{i, j\} = \emptyset$, we have

$$
\begin{aligned}
&(14_B) \wedge (10_B) : w = \mu_{ki}^{-1}\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{a_1a_2})^{y}\mu_{ij},\\
&(f,g)_w = [\mu_{ki}^{-1}\mu_{ki}-1]\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{a_1a_2})^{y}\mu_{ij}\\
&-\mu_{ki}^{-1}[\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{a_1a_2})^{y}\mu_{ij}-\mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}\\
&(\mu_{z_3z_1}\mu_{z_2z_1})^{x}(\mu_{a_1a_2})^{y}] = \mu_{ki}^{-1}\mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}(\mu_{z_3z_1}\mu_{z_2z_1})^{x}(\mu_{a_1a_2})^{y}\\
&-\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{a_1a_2})^{y}\mu_{ij}\\
&\equiv \mu_{ki}^{-1}\mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}(\mu_{z_3z_1}\mu_{z_2z_1})^{x}(\mu_{a_1a_2})^{y}-\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}\\
&(\mu_{a_1a_2})^{y}\mu_{ij}\\
&\equiv \mu_{ki}\mu_{ki}^{-1}\mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}(\mu_{z_3z_1}\mu_{z_2z_1})^{x}(\mu_{a_1a_2})^{y}-\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}\\
&(\mu_{z_2z_1}\mu_{z_3z_1})^{x}(\mu_{a_1a_2})^{y}\mu_{ij}\\
&\equiv \mu_{ij}\mu_{kj}\mu_{ki}\mu_{il}\mu_{jl}(\mu_{z_3z_1}\mu_{z_2z_1})^{x}(\mu_{a_1a_2})^{y}-\mu_{ki}\mu_{kj}\mu_{jl}\mu_{il}(\mu_{z_2z_1}\mu_{z_3z_1})^{x}\\
&(\mu_{a_1a_2})^{y}\mu_{ij}\equiv 0.
\end{aligned}
$$

Since there are no any inclusion compositions of relations $(1_A) - (14_B)$, the proof is completed. □

Now let R be the set of relations for pure virtual braid group PV_3 given in Theorem 2.1. Let $C(u)$ be normal form of a word $u \in PV_3$. By using the Composition-Diamond Lemma 1.1 and Theorem 2.1, the normal form for elements of pure virtual braid group PV_3 can be given as follows.

Corollary 2.3. *Let $u \in PV_3$. Then $C(u)$ consists of one of the following conditions:*
(1) The normal form is of the form

$$(\mu_{r_ks_k})^{\alpha_1}(\mu_{r_js_k})^{\alpha_2}(\mu_{r_jr_k})^{\alpha_3}W_{rs}$$

such that either $r_k < s_k < r_j$ *or* $s_k < r_k < r_j$ *or* $r_k < r_j < s_k$.
(2) The normal form is of the form

$$(\mu_{r_k s_k})^{\alpha_1}(\mu_{r_k s_j})^{\alpha_2}(\mu_{s_k s_j})^{\alpha_3} W_{rs}$$

such that either $r_k < s_k < s_j$ *or* $r_k < s_j < s_k$ *or* $s_j < r_k < s_k$.
The word W_{rs} *is an irreducible word in* PV_3, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ *and* $1 \leq r_k, s_k, r_j, s_j \leq 3$.

By considering relations given in Theorem 2.2, it is quite diffucult to find a specific normal form structure for elements of pure virtual braid group PV_n $(n \geq 4)$. Because of this, we partially give normal form for elements of PV_n $(n \geq 4)$ as follows. Let R be the set of relations for pure virtual braid group PV_n $(n \geq 4)$ given in Theorem 2.2. Let $C(u)$ be normal form of a word $u \in PV_n$.

COROLLARY 2.4. *Let* $u \in PV_n$ $(n \geq 4)$. *Then* $C(u)$ *is of the form* $(\mu_{r_1 s_1})^{\alpha_1}(\mu_{r_2 s_2})^{\alpha_2} W_{rs}$, *where the word* W_{rs} *is an irreducible word in* PV_n *and* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$. *We also have the following:*
(1) If $r_1 = r_2$, *either* $s_1 > r_1 = r_2 > s_2$ *or* $s_1 > s_2 > r_1 = r_2$ *or the normal form is of the form* $(\mu_{r_1 s_1})^{\alpha_1}(\mu_{r_2 s_2})^{\alpha_2}(\mu_{r_3 s_3})^{\alpha_3} W_{rs}$ *such that* $s_2 = s_3$ *and* $r_3 > r_2$.
(2) If $s_1 = s_2$, *then* $r_2 > r_1$.
(3) If $s_1 = r_2$, *then* $s_1 = r_2 > r_1 > s_2$.
(4) If $\{r_1, r_2\} \cap \{s_1, s_2\} = \emptyset$, *then* $r_2 > r_1$.

Finally, we note that by Corollary 2.3 and Corollary 2.4 we can say the well known fact that the word problem is solvable for pure virtual braid groups.

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REFERENCES

1. S.I. ADIAN AND V.G. DURNEV, Decision problems for groups and semigroups, *Russ. Math. Surv.* **55**(2) (2000), 207–296.
2. U. ALI AND B. BERCEANU, Canonical forms of positive braids, *J. Algebra Appl.* **14**(1) (2015), 1450076.
3. F. ATEŞ, A.S. ÇEVIK, AND E.G. KARPUZ, Gröbner-Shirshov basis for the singular part of the Brauer semigroup, *Turk. J. Math.* **42** (2018), 1338–1347.
4. F. ATEŞ, E.G. KARPUZ, C. KOCAPINAR, AND A.S. ÇEVIK, Gröbner-Shirshov bases of some monoids, *Discrete Math.* **311** (2011), 1064–1071.
5. V.G. BARDAKOV, The virtual and universal braids, *Fund. Math.* **181** (2004), 1–18.
6. V.G. BARDAKOV, R. MIKHAILOV, V.V. VERSHININ, AND J. WU, On the pure virtual braid PV_3, *Commun. Algebra* **44**(3) (2016), 1350–1378.
7. L. BARTHOLDI, B. ENRIQUEZ, P. ETINGOF, AND E. RAINS, Groups and Lie algebras corresponding to the Yang-Baxter equations, *J. Algebra* **305** (2006), 742–764.

8. P. BELLINGERI, B. CISNEROS DE LA CRUZ, AND L. PARIS, A simple solution to the word problem for virtual braid groups, *Pac. J. Math.* **283**(2) (2015), 271–287.

9. G.M. BERGMAN, The diamond lemma for ring theory, *Adv. Math.* **29** (1978), 178–218.

10. L.A. BOKUT, Unsolvability of the word problem and subalgebras of finitely presented Lie algebras, *Math. USSR Izv.* **36** (1972), 1173–1219.

11. ————————, Imbedding into simple associative algebras, *Algebr. Log.* **15** (1976), 117–142.

12. ————————, Gröbner-Shirshov basis for the Braid group in the Artin-Garside generators, *J. Symb. Comput.* **43** (2008), 397–405.

13. ————————, Gröbner-Shirshov basis for the Braid group in the Birman Ko-Lee generators, *J. Algebra* **321** (2009), 361–376.

14. L.A. BOKUT AND Y. CHEN, Gröbner-Shirshov bases and their calculation, *B. Math. Sci.* **4** (2014), 325–395.

15. L.A. BOKUT AND A. VESNIN, Gröbner-Shirshov bases for some Braid groups, *J. Symb. Comput.* **41** (2006), 357–371.

16. B. BUCHBERGER, *An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal,* (in German), Ph.D., University of Innsbruck, Innsbruck, Austria, 1965.

17. ————————, An algorithmical criteria for the solvability of algebraic systems of equations, (in German), *Aequationes Math.* **4** (1970), 374–383.

18. Y. CHEN AND C. ZHONG, Gröbner-Shirshov bases for braid groups in Adyan-Thurston generators, *Algebr. Colloq.* **20**(2) (2013), 309–318.

19. E.G. KARPUZ, Gröbner-Shirshov bases of some semigroup constructions, *Algebr. Colloq.* **22**(1) (2015), 35–46.

20. E.G. KARPUZ, F. ATEŞ, AND A.S. ÇEVIK, Gröbner-Shirshov bases of some Weyl groups, *Rocky Mt. J. Math.* **45**(4) (2015), 1165–1175.

21. E.G. KARPUZ, F. ATEŞ, N. URLU, A.S. ÇEVIK, AND I.N. CANGÜL, A note on the Gröbner-Shirshov bases over ad-hoc extensions of groups, *Filomat* **30**(4) (2016), 1037–1043.

22. L.H. KAUFFMAN, Virtual knot theory, *Eur. J. Combin.* **20** (1999), 663–690.

23. L.H. KAUFFMAN AND S. LAMBROPOULOU, A categorical structure for the virtual braid group, *Commun. Algebra* **39**(12) (2011), 4679–4704.

24. C. KOCAPINAR, E.G. KARPUZ, F. ATEŞ, AND A.S. ÇEVIK, Gröbner-Shirshov bases of the generalized Bruck-Reilly ∗-extension, *Algebr. Colloq.* **19**(1) (2012), 813–820.

25. A.I. SHIRSHOV, Some algorithmic problems for Lie algebras, *Siberian Math. J.* **3** (1962), 292–296.

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