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Article *in* Asian-European Journal of Mathematics · July 2019

DOI: 10.1142/S1793557120400136

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Asian-European Journal of Mathematics
Vol. 12, No. 6 (2019) 2040013 (30 pages)

DOI: 10.1142/S1793557120400136

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Communicated by J. Koppitz
Received July 11, 2019
Accepted July 11, 2019
Published August 13, 2019

The aim of this paper is to obtain the solvability of the word problem over congruence classes of complex reflection groups G_{24} and G_7. To do that, we use Gröbner–Shirshov basis theory and get the normal form structure of elements of these group types.

Keywords: Complex reflection group; Gröbner–Shirshov basis; normal form; word problem.

AMS Subject Classification: 16S15, 20F05, 20F36, 20E22, 20M05

1. Introduction and Preliminaries

It is well known that a complex reflection group is a finite group acting on a finite-dimensional complex vector space that is generated by complex reflection. In 1954, Shephard and Todd classified all the finite complex reflection groups in their paper [19]. A finite complex reflection group G can be presented by generators and related relations as well-known Coxeter group. From the point of combinatorial group theory, such a presentation is not unique for G in general. So different presentations of a finite reflection group G may reveal various different and important properties of G [20]. For this reason, it is worth studying the congruence relations of complex reflection groups. In [21], Shi introduced two concepts for any complex reflection group G generated by more than two reflections: one of them is the equivalence of simple root systems, and the other one is the congruence classes of presentations. According to Shi's definition, the equivalent simple root systems of G determine the congruent presentations of G. In [21, 24, 25], authors have obtained the presentations for congruence classes of complex reflection groups G_{24} and G_7. In this paper, by considering the presentations given in [21, 24, 25], we study whether or not the word problem for congruence classes of complex reflection groups G_{24} and G_7 can be solved by using Gröbner–Shirshov basis theory. It is well known that *word problem* is one and the most popular of the decision problems which ask for a yes or no answer to a specific question [1]. This problem has been introduced by Max Dehn in early 1900's and it is not solvable in general for finitely presented groups, that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group.

In general, for Coxeter groups, the word problem is rather hard to compute. Since finite reflection groups are a kind of Coxeter group type, it is worth studying whether or not the word problem for congruence classes of complex reflection groups G_{24} and G_7 can be solved. The method of Gröbner–Shirshov basis which is the main theme of this paper gives a new algorithm to get normal forms of elements of groups, and so a new algorithm for solving the word problem in these groups.

Throughout this paper, the order of words will be chosen in the given alphabet in the meaning of deg-lex way comparing two words first by their lengths and then lexicographically when the lengths are equal. Additionally the notations $(i) \wedge (j)$ will denote the intersection composition of relations (i) and (j). We finally note that some background, historical material and some results on Gröbner–Shirshov basis theory can be found, for instance, in [2–4, 6–9, 12–18].

1.1. *Gröbner–Shirshov basis theory*

Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity denoted by 1. Suppose that X^* is a well ordered monoid. A well ordering $<$ on X^* is called *monomial* if, for $u, v \in X^*$, we have $u < v \Rightarrow w_1uw_2 < w_1vw_2$, for all $w_1, w_2 \in X^*$. Every nonzero polynomial $f \in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Let f and g be two monic polynomials in $K\langle X\rangle$. We then have two compositions as follows:

- If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.

- If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

If g is monic, $\overline{f} = a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then transformation $f \mapsto f - \alpha agb$ is called elimination (ELW) of the leading word of g in f.

Let $S \subseteq K\langle X\rangle$ with each $s \in S$ is monic. Then the composition $(f, g)_w$ is called trivial modulo (S, w) if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K, a_i, b_i \in X^*, s_i \in S$ and $a_i\overline{s_i}b_i < w$. If this is the case, then we write $(f, g)_w \equiv 0 \bmod(S, w)$.

We call the set S endowed with the well ordering $<$ a *Gröbner–Shirshov basis* for $K\langle X \mid S\rangle$ if any composition $(f, g)_w$ of polynomials in S is trivial modulo S and corresponding w.

The following lemma was proved by Shirshov [23] for free Lie algebras with deg-lex ordering.

Lemma 1.1 (Composition-Diamond Lemma). *Let K be a field, $A = K\langle X \mid S\rangle = K\langle X\rangle/\mathrm{Id}(S)$ and $<$ a monomial ordering on X^*, where $\mathrm{Id}(S)$ is the ideal of $K\langle X\rangle$ generated by S. Then the following statements are equivalent:*

(1) *S is a Gröbner–Shirshov basis.*
(2) *$f \in \mathrm{Id}(S) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in S$ and $a, b \in X^*$.*
(3) *$\mathrm{Irr}(S) = \{u \in X^* \mid u \neq a\overline{s}b, s \in S, a, b \in X^*\}$ is a basis for the algebra $A = K\langle X \mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner–Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process many times (maybe infinitely), we eventually obtain a Gröbner–Shirshov basis S^{comp}. We should note that such a process is called the *Shirshov algorithm*. In fact, this lemma is called the Composition-Diamond Lemma (or Buchberger's Theorem in some sources) and different versions of the proof of it can be found in [4, 6, 8, 23].

1.2. *Congruence classes of complex reflection groups G_{24} and G_7*

In [19], the authors have classified all finite complex reflection groups. Later, Cohen [10] gave a more systematic description for these groups in terms of root systems, vector graphs and root graphs [10]. Recently, in [11], Howlett and Shi defined a simple root system (B, w) for such these groups which is analogous to the corresponding concept for a Coxeter group. Finite complex reflection groups are divided into two main classes: primitive and imprimitive. Any imprimitive complex reflection group has the form $G(m, p, n)$ for some positive integers m, p, n with $p \mid m$, $m > 2, n > 1$, and $(m, p, n) \neq (m, m, 2)$ [10]. The imprimitive complex reflection groups form an infinite series. There are 23 primitive complex reflection groups in total, 8 of them has exactly one congruence class of presentations since they can be generated by only two reflections [5, 22]. In this paper, we have worked on two congruence classes, namely G_{24} and G_7.

2. Results

2.1. *A Gröbner–Shirshov basis for congruence class of complex reflection group G_{24}*

The congruence class of complex reflection group G_{24} is generated by three reflections of order two. The monoid presentation of this group is given as follows [22].

$$\begin{aligned}\mathcal{P}_{G_{24}} = \langle s, t, u, s^{-1}, t^{-1}, u^{-1};\ & s^2 = t^2 = u^2 = 1, stst = tsts, sus = usu,\\ & utut = tutu, tusutu = usutus, ss^{-1} = s^{-1}s = 1,\\ & tt^{-1} = t^{-1}t = 1, uu^{-1} = u^{-1}u = 1\rangle. \end{aligned} \tag{2.1}$$

Theorem 2.1. *The congruence class of complex reflection group G_{24} has a Gröbner–Shirshov basis with respect to the degree-lexicographic order $u^{-1} > t^{-1} > s^{-1} > u > t > s$ as follows*:

(1) $s^{-1} = s$,	(2) $t^{-1} = t$,
(3) $u^{-1} = u$,	(4) $s^2 = 1$,
(5) $t^2 = 1$,	(6) $u^2 = 1$,
(7) $usu = sus$,	(8) $(ts)^2 = (st)^2$,
(9) $(ut)^2 = (tu)^2$,	(10) $tsusts = sustsu$
(11) $tsustu = sustus$,	(12) $ustsus = stsust$,
(13) $us(tu)^2 = sustut$,	(14) $utsust = sutsus$,
(15) $tstustu = stustsu$,	(16) $t(stu)^2 = (stu)^2s$,
(17) $ustsuts = stsutst$,	(18) $(ust)^2u = (stu)^2t$,
(19) $(uts)^2t = (sut)^2s$,	(20) $(uts)^2u = t(uts)^2$,
(21) $utsutus = tutsutu$,	(22) $utu(st)^2 = (tu)^2sts$,
(23) $utustut = sutustu$,	(24) $tustutsu = sutstust$,
(25) $(tuts)^2 = (stut)^2$,	(26) $ustutsus = sustutsu$,
(27) $ustutsut = tsutstus$,	(28) $(utst)^2 = (tstu)^2$,
(29) $(tstu)^2t = utstuts$,	(30) $tsutstust = ustutsu$,
(31) $tsutstuts = stsutstut$,	(32) $(tus)^2tsu = stutstust$,
(33) $tustutstu = su(tus)^2t$,	(34) $(ust)^2sut = (tstu)^2s$,
(35) $u(stut)^2 = tsu(tus)^2$,	(36) $utstutsut = tstutsutu$,
(37) $(tstu)^2st = (ust)^2su$,	(38) $tsu(tus)^2t = ustutstu$,

(39) $tu(st)^2utsu = suts(tu)^2st$, (40) $(tus)^2tstu = stu(tus)^2t$,

(41) $us(tstu)^2 = su(st)^2utst$, (42) $u(st)^2utsut = tsuts(tu)^2s$,

(43) $ustsutusts = stsutustsu$, (44) $ustsutustu = stsu(tus)^2$,

(45) $(ust)^2stut = tstu(tus)^2$, (46) $ustutstust = tu(tus)^2ts$,

(47) $uts(tu)^2sts = tuts(tu)^2st$, (48) $uts(tu)^2stu = stu(st)^2uts$,

(49) $utustsutus = sutustsutu$, (50) $u(tus)^2tst = su(tus)^2ts$,

(51) $tstu(tus)^2t = (ust)^2stu$, (52) $tsuts(tu)^2st = u(st)^2utsu$,

(53) $u(st)^2utusts = stsutustsut$, (54) $u(st)^2utustu = stu(st)^2utst$,

(55) $u(stust)^2 = s(ust)^2stus$.

Proof. Firstly, we see that all relations given in the presentation (2.1) hold in Gröbner–Shirshov basis given above. Now we need to prove that all compositions among relations (1)–(55) are trivial. To do that, we have the following ambiguities.

$(4) \wedge (4) : w = s^3$, $(5) \wedge (5) : w = t^3$,

$(5) \wedge (8) : w = t^2sts$, $(5) \wedge (10) : w = t^2susts$,

$(5) \wedge (11) : w = t^2sustu$, $(5) \wedge (15) : w = t^2stustu$,

$(5) \wedge (16) : w = t^2(stu)^2$, $(5) \wedge (24) : w = t^2ustutsu$,

$(5) \wedge (25) : w = t^2utstuts$, $(5) \wedge (29) : w = t^2stutstut$,

$(5) \wedge (30) : w = t^2sutstust$, $(5) \wedge (31) : w = t^2sutstuts$,

$(5) \wedge (32) : w = t^2ustustsu$, $(5) \wedge (33) : w = t^2ustutstu$,

$(5) \wedge (37) : w = t^2stutstust$, $(5) \wedge (38) : w = t^2su(tus)^2t$,

$(5) \wedge (39) : w = t^2u(st)^2utsu$, $(5) \wedge (40) : w = t^2ustuststu$,

$(5) \wedge (51) : w = t^2stu(tus)^2t$, $(5) \wedge (52) : w = t^2suts(tu)^2st$,

$(6) \wedge (6) : w = u^3$, $(6) \wedge (7) : w = u^2su$,

$(6) \wedge (9) : w = u^2tut$, $(6) \wedge (12) : w = u^2stsus$,

$(6) \wedge (13) : w = u^2s(tu)^2$, $(6) \wedge (14) : w = u^2tsust$,

$(6) \wedge (17) : w = u^2stsuts$, $(6) \wedge (18) : w = u^2stustu$,

$(6) \wedge (19) : w = u^2tsutst$, $(6) \wedge (20) : w = u^2tsutsu$,

$(6) \wedge (21) : w = u^2tsutus$, $(6) \wedge (22) : w = u^2tu(st)^2$,

$(6) \wedge (23) : w = u^2tustut$, $(6) \wedge (26) : w = u^2stutsus$,

$(6) \wedge (27) : w = u^2stutsut$, $(6) \wedge (28) : w = u^2tstutst$,

$(6) \wedge (34) : w = u^2stustsut$, $(6) \wedge (35) : w = u^2(stut)^2$,

$(6) \wedge (36) : w = u^2tstutsut$, $(6) \wedge (41) : w = u^2s(tstu)^2$,

$(6) \wedge (42) : w = u^2(st)^2utsut$, $(6) \wedge (43) : w = u^2stsutusts$,

$(6) \wedge (44) : w = u^2 stsutustu,$ $(6) \wedge (45) : w = u^2 stuststut,$
$(6) \wedge (46) : w = u^2 stutstust,$ $(6) \wedge (47) : w = u^2 ts(tu)^2 sts,$
$(6) \wedge (48) : w = u^2 ts(tu)^2 stu,$ $(6) \wedge (49) : w = u^2 tustsutus,$
$(6) \wedge (50) : w = u^2 (tus)^2 tst,$ $(6) \wedge (53) : w = u^2 (st)^2 utusts,$
$(6) \wedge (54) : w = u^2 (st)^2 utustu,$ $(6) \wedge (55) : w = u^2 (stust)^2,$
$(7) \wedge (6) : w = usu^2,$ $(7) \wedge (7) : w = ususu,$
$(7) \wedge (9) : w = usutut,$ $(7) \wedge (12) : w = usustsus,$
$(7) \wedge (13) : w = usus(tu)^2,$ $(7) \wedge (14) : w = usutsust,$
$(7) \wedge (17) : w = usustsuts,$ $(7) \wedge (18) : w = usustustu,$
$(7) \wedge (19) : w = usutsutst,$ $(7) \wedge (20) : w = usutsutsu,$
$(7) \wedge (21) : w = usutsutus,$ $(7) \wedge (22) : w = usutu(st)^2,$
$(7) \wedge (23) : w = usutustut,$ $(7) \wedge (26) : w = usustutsus,$
$(7) \wedge (27) : w = usustutsut,$ $(7) \wedge (28) : w = usutstutst,$
$(7) \wedge (34) : w = usustustsut,$ $(7) \wedge (35) : w = usu(stut)^2,$
$(7) \wedge (36) : w = usutstutsut,$ $(7) \wedge (41) : w = usus(tstu)^2,$
$(7) \wedge (42) : w = usu(st)^2 utsut,$ $(7) \wedge (43) : w = usustsutusts,$
$(7) \wedge (44) : w = usustsutustu,$ $(7) \wedge (45) : w = usustuststut,$
$(7) \wedge (46) : w = usustutstust,$ $(7) \wedge (47) : w = usuts(tu)^2 sts,$
$(7) \wedge (48) : w = usuts(tu)^2 stu,$ $(7) \wedge (49) : w = usutustsutus,$
$(7) \wedge (50) : w = usu(tus)^2 tst,$ $(7) \wedge (53) : w = usu(st)^2 utusts,$
$(7) \wedge (54) : w = usu(st)^2 utustu,$ $(7) \wedge (55) : w = usu(stust)^2,$
$(8) \wedge (8) : w = (ts)^3,$ $(8) \wedge (15) : w = tststustu,$
$(8) \wedge (16) : w = tst(stu)^2,$ $(8) \wedge (29) : w = ts(tstu)^2 t,$
$(8) \wedge (30) : w = tstsutstust,$ $(8) \wedge (31) : w = tstsutstuts,$
$(8) \wedge (37) : w = ts(tstu)^2 st,$ $(8) \wedge (38) : w = tstsu(tus)^2 t,$
$(8) \wedge (51) : w = tststu(tus)^2 t,$ $(8) \wedge (52) : w = tstsuts(tu)^2 st,$
$(9) \wedge (5) : w = utut^2,$ $(9) \wedge (8) : w = utu(ts)^2,$
$(9) \wedge (10) : w = ututsusts,$ $(9) \wedge (11) : w = ututsustu,$
$(9) \wedge (15) : w = ututstustu,$ $(9) \wedge (16) : w = utut(stu)^2,$
$(9) \wedge (24) : w = ututustutsu,$ $(9) \wedge (25) : w = utu(tuts)^2,$
$(9) \wedge (29) : w = utu(tstu)^2 t,$ $(9) \wedge (30) : w = ututsutstust,$
$(9) \wedge (31) : w = ututsutstuts,$ $(9) \wedge (32) : w = utu(tus)^2 tsu,$
$(9) \wedge (33) : w = ututustutstu,$ $(9) \wedge (37) : w = utu(tstu)^2 st,$

$(9) \wedge (38) : w = ututsu(tus)^2t,$ $(9) \wedge (39) : w = ututu(st)^2utsu,$
$(9) \wedge (40) : w = utu(tus)^2tstu,$ $(9) \wedge (51) : w = ututstu(tus)^2t,$
$(9) \wedge (52) : w = ututsuts(tu)^2st,$ $(10) \wedge (8) : w = tsus(ts)^2,$
$(10) \wedge (15) : w = tsuststustu,$ $(10) \wedge (16) : w = tsust(stu)^2,$
$(10) \wedge (29) : w = tsus(tstu)^2t,$ $(10) \wedge (30) : w = tsustsutstust,$
$(10) \wedge (31) : w = tsustsutstuts,$ $(10) \wedge (37) : w = tsus(tstu)^2st,$
$(10) \wedge (38) : w = tsustsu(tus)^2t,$ $(10) \wedge (51) : w = tsuststu(tus)^2t,$
$(10) \wedge (52) : w = tsustsuts(tu)^2st,$ $(11) \wedge (6) : w = tsutu^2,$
$(11) \wedge (7) : w = tsutusu,$ $(11) \wedge (9) : w = tsututut,$
$(11) \wedge (12) : w = tsutustsus,$ $(11) \wedge (13) : w = tsutus(tu)^2,$
$(11) \wedge (14) : w = tsututsust,$ $(11) \wedge (17) : w = tsutustsuts,$
$(11) \wedge (18) : w = tsut(ust)^2u,$ $(11) \wedge (19) : w = tsut(uts)^2t,$
$(11) \wedge (20) : w = tsut(uts)^2u,$ $(11) \wedge (21) : w = tsututsutus,$
$(11) \wedge (22) : w = tsututu(st)^2,$ $(11) \wedge (23) : w = tsututustut,$
$(11) \wedge (26) : w = tsutustutsus,$ $(11) \wedge (27) : w = tsutustutsut,$
$(11) \wedge (28) : w = tsut(utst)^2,$ $(11) \wedge (34) : w = tsut(ust)^2sut,$
$(11) \wedge (35) : w = tsutu(stut)^2,$ $(11) \wedge (36) : w = tsututstutsut,$
$(11) \wedge (41) : w = tsutus(tstu)^2,$ $(11) \wedge (42) : w = tsutu(st)^2utsut,$
$(11) \wedge (43) : w = tsutustsutusts,$ $(11) \wedge (44) : w = tsutustsutustu,$
$(11) \wedge (45) : w = tsut(ust)^2stut,$ $(11) \wedge (46) : w = tsutustutstust,$
$(11) \wedge (47) : w = tsututs(tu)^2sts,$ $(11) \wedge (48) : w = tsututs(tu)^2stu,$
$(11) \wedge (49) : w = tsututustsutus,$ $(11) \wedge (50) : w = tsutu(tus)^2tst,$
$(11) \wedge (53) : w = tsutu(st)^2utusts,$ $(11) \wedge (54) : w = tsutu(st)^2utustu,$
$(11) \wedge (55) : w = tsutu(stust)^2,$ $(12) \wedge (7) : w = ustsusu,$
$(12) \wedge (12) : w = ustsustsus,$ $(12) \wedge (13) : w = ustsus(tu)^2,$
$(12) \wedge (17) : w = ustsustsuts,$ $(12) \wedge (18) : w = usts(ust)^2u,$
$(12) \wedge (26) : w = ustsustutsus,$ $(12) \wedge (27) : w = ustsustutsut,$
$(12) \wedge (34) : w = usts(ust)^2sut,$ $(12) \wedge (35) : w = ustsu(stut)^2,$
$(12) \wedge (41) : w = ustsus(tstu)^2,$ $(12) \wedge (42) : w = ustsu(st)^2utsut,$
$(12) \wedge (43) : w = ustsustsutusts,$ $(12) \wedge (44) : w = ustsustsutustu,$
$(12) \wedge (45) : w = usts(ust)^2stut,$ $(12) \wedge (46) : w = ustsustutstust,$
$(12) \wedge (53) : w = ustsu(st)^2utusts,$ $(12) \wedge (54) : w = ustsu(st)^2utustu,$
$(12) \wedge (55) : w = ustsu(stust)^2,$ $(13) \wedge (6) : w = ustutu^2,$

$(13) \wedge (7) : w = ustutusu,$ $(13) \wedge (9) : w = ustututut,$
$(13) \wedge (12) : w = ustutustsus,$ $(13) \wedge (13) : w = ustutus(tu)^2,$
$(13) \wedge (14) : w = ustututsust,$ $(13) \wedge (17) : w = ustutustsuts,$
$(13) \wedge (18) : w = ustut(ust)^2u,$ $(13) \wedge (19) : w = ustut(uts)^2t,$
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$(13) \wedge (22) : w = ustututu(st)^2,$ $(13) \wedge (23) : w = ustututustut,$
$(13) \wedge (26) : w = ustutustutsus,$ $(13) \wedge (27) : w = ustutustutsut,$
$(13) \wedge (28) : w = ustut(utst)^2,$ $(13) \wedge (34) : w = ustut(ust)^2sut,$
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$(13) \wedge (47) : w = ustututs(tu)^2sts,$ $(13) \wedge (48) : w = ustututs(tu)^2stu,$
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$(13) \wedge (53) : w = ustutu(st)^2utusts,$ $(13) \wedge (54) : w = ustutu(st)^2utustu,$
$(13) \wedge (55) : w = ustutu(stust)^2,$ $(14) \wedge (5) : w = utsust^2,$
$(14) \wedge (8) : w = utsus(ts)^2,$ $(14) \wedge (10) : w = utsustsusts,$
$(14) \wedge (11) : w = utsustsustu,$ $(14) \wedge (15) : w = utsuststustu,$
$(14) \wedge (16) : w = utsust(stu)^2,$ $(14) \wedge (24) : w = utsustustutsu,$
$(14) \wedge (25) : w = utsus(tuts)^2,$ $(14) \wedge (29) : w = utsus(tstu)^2t,$
$(14) \wedge (30) : w = utsustsutstust,$ $(14) \wedge (31) : w = utsustsutstuts,$
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$(14) \wedge (51) : w = utsuststu(tus)^2t,$ $(14) \wedge (52) : w = utsustsuts(tu)^2st,$
$(15) \wedge (6) : w = tstustu^2,$ $(15) \wedge (7) : w = tstustusu,$
$(15) \wedge (9) : w = tstustutut,$ $(15) \wedge (12) : w = tstustustsus,$
$(15) \wedge (13) : w = tstustus(tu)^2,$ $(15) \wedge (14) : w = tstustutsust,$
$(15) \wedge (17) : w = tstustustsuts,$ $(15) \wedge (18) : w = tstust(ust)^2u,$
$(15) \wedge (19) : w = tstust(uts)^2t,$ $(15) \wedge (20) : w = tstust(uts)^2u,$
$(15) \wedge (21) : w = tstustutsutus,$ $(15) \wedge (22) : w = tstustutu(st)^2,$
$(15) \wedge (23) : w = tstustutustut,$ $(15) \wedge (26) : w = tstustustutsus,$
$(15) \wedge (27) : w = tstustustutsut,$ $(15) \wedge (28) : w = tstust(utst)^2,$

$(15) \wedge (34) : w = tstust(ust)^2sut,$ $(15) \wedge (35) : w = tstustu(stut)^2,$
$(15) \wedge (36) : w = tstustutstutsut,$ $(15) \wedge (41) : w = tstustus(tstu)^2,$
$(15) \wedge (42) : w = tstustu(st)^2utsut,$ $(15) \wedge (43) : w = tstustustsutusts,$
$(15) \wedge (44) : w = tstustustsutustu,$ $(15) \wedge (45) : w = tstust(ust)^2stut,$
$(15) \wedge (46) : w = tstustustutstust,$ $(15) \wedge (47) : w = tstustuts(tu)^2sts,$
$(15) \wedge (48) : w = tstustuts(tu)^2stu,$ $(15) \wedge (49) : w = tstustutustsutus,$
$(15) \wedge (50) : w = tstustu(tus)^2tst,$ $(15) \wedge (53) : w = tstustu(st)^2utusts,$
$(15) \wedge (54) : w = tstustu(st)^2utustu,$ $(15) \wedge (55) : w = tstustu(stust)^2,$
$(16) \wedge (6) : w = tstustu^2,$ $(16) \wedge (7) : w = tstustusu,$
$(16) \wedge (9) : w = tstustutut,$ $(16) \wedge (12) : w = tstustustsus,$
$(16) \wedge (13) : w = tstustus(tu)^2,$ $(16) \wedge (14) : w = tstustutsust,$
$(16) \wedge (17) : w = tstustustsuts,$ $(16) \wedge (18) : w = tstust(ust)^2u,$
$(16) \wedge (19) : w = tstust(uts)^2t,$ $(16) \wedge (20) : w = tstust(uts)^2u,$
$(16) \wedge (21) : w = tstustutsutus,$ $(16) \wedge (22) : w = tstustutu(st)^2,$
$(16) \wedge (23) : w = tstustutustut,$ $(16) \wedge (26) : w = tstustustutsus,$
$(16) \wedge (27) : w = tstustustutsut,$ $(16) \wedge (28) : w = tstust(utst)^2,$
$(16) \wedge (34) : w = tstust(ust)^2sut,$ $(16) \wedge (35) : w = tstustu(stut)^2,$
$(16) \wedge (36) : w = tstustutstutsut,$ $(16) \wedge (41) : w = tstustus(tstu)^2,$
$(16) \wedge (42) : w = tstustu(st)^2utsut,$ $(16) \wedge (43) : w = tstustustsutusts,$
$(16) \wedge (44) : w = tstustustsutustu,$ $(16) \wedge (45) : w = tstust(ust)^2stut,$
$(16) \wedge (46) : w = tstustustutstust,$ $(16) \wedge (47) : w = tstustuts(tu)^2sts,$
$(16) \wedge (48) : w = tstustuts(tu)^2stu,$ $(16) \wedge (49) : w = tstustutustsutus,$
$(16) \wedge (50) : w = tstustu(tus)^2tst,$ $(16) \wedge (53) : w = tstustu(st)^2utusts,$
$(16) \wedge (54) : w = tstustu(st)^2utustu,$ $(16) \wedge (55) : w = tstustu(stust)^2,$
$(17) \wedge (8) : w = ustsu(ts)^2,$ $(17) \wedge (15) : w = ustsutstustu,$
$(17) \wedge (16) : w = ustsut(stu)^2,$ $(17) \wedge (29) : w = ustsu(tstu)^2t,$
$(17) \wedge (30) : w = ustsutsutstust,$ $(17) \wedge (31) : w = ustsutsutstuts,$
$(17) \wedge (37) : w = ustsu(tstu)^2st,$ $(17) \wedge (38) : w = ustsutsu(tus)^2t,$
$(17) \wedge (51) : w = ustsutstu(tus)^2t,$ $(17) \wedge (52) : w = ustsutsuts(tu)^2st,$
$(18) \wedge (6) : w = (ust)^2u^2,$ $(18) \wedge (7) : w = (ust)^2usu,$
$(18) \wedge (9) : w = (ust)^2utut,$ $(18) \wedge (12) : w = (ust)^2ustsus,$
$(18) \wedge (13) : w = (ust)^2us(tu)^2,$ $(18) \wedge (14) : w = (ust)^2utsust,$
$(18) \wedge (17) : w = (ust)^2ustsuts,$ $(18) \wedge (18) : w = (ust)^2(ust)^2u,$

$(18)\wedge(19): w=(ust)^2(uts)^2t$, $(18)\wedge(20): w=(ust)^2(uts)^2u$,
$(18)\wedge(21): w=(ust)^2utsutus$, $(18)\wedge(22): w=(ust)^2utu(st)^2$,
$(18)\wedge(23): w=(ust)^2utustut$, $(18)\wedge(26): w=(ust)^2ustutsus$,
$(18)\wedge(27): w=(ust)^2ustutsut$, $(18)\wedge(28): w=(ust)^2(utst)^2$,
$(18)\wedge(34): w=(ust)^2(ust)^2sut$, $(18)\wedge(35): w=(ust)^2u(stut)^2$,
$(18)\wedge(36): w=(ust)^2utstutsut$, $(18)\wedge(41): w=(ust)^2us(tstu)^2$,
$(18)\wedge(42): w=(ust)^2u(st)^2utsut$, $(18)\wedge(43): w=(ust)^2ustsutusts$,
$(18)\wedge(44): w=(ust)^2ustsutustu$, $(18)\wedge(45): w=(ust)^2(ust)^2stut$,
$(18)\wedge(46): w=(ust)^2ustutstust$, $(18)\wedge(47): w=(ust)^2uts(tu)^2sts$,
$(18)\wedge(48): w=(ust)^2uts(tu)^2stu$, $(18)\wedge(49): w=(ust)^2utustsutus$,
$(18)\wedge(50): w=(ust)^2u(tus)^2tst$, $(18)\wedge(53): w=(ust)^2u(st)^2utusts$,
$(18)\wedge(54): w=(ust)^2u(st)^2utustu$, $(18)\wedge(55): w=(ust)^2u(stust)^2$,
$(19)\wedge(5): w=(uts)^2t^2$, $(19)\wedge(8): w=(uts)^2(ts)^2$,
$(19)\wedge(10): w=(uts)^2tsusts$, $(19)\wedge(11): w=(uts)^2tsustu$,
$(19)\wedge(15): w=(uts)^2tstustu$, $(19)\wedge(16): w=(uts)^2t(stu)^2$,
$(19)\wedge(24): w=(uts)^2tustutsu$, $(19)\wedge(25): w=(uts)^2(tuts)^2$,
$(19)\wedge(29): w=(uts)^2(tstu)^2t$, $(19)\wedge(30): w=(uts)^2tsutstust$,
$(19)\wedge(31): w=(uts)^2tsutstuts$, $(19)\wedge(32): w=(uts)^2(tus)^2tsu$,
$(19)\wedge(33): w=(uts)^2tustutstu$, $(19)\wedge(37): w=(uts)^2(tstu)^2st$,
$(19)\wedge(38): w=(uts)^2tsu(tus)^2t$, $(19)\wedge(39): w=(uts)^2tu(st)^2utsu$,
$(19)\wedge(40): w=(uts)^2(tus)^2tstu$, $(19)\wedge(51): w=(uts)^2tstu(tus)^2t$,
$(19)\wedge(52): w=(uts)^2tsuts(tu)^2st$, $(20)\wedge(6): w=(uts)^2u^2$,
$(20)\wedge(7): w=(uts)^2usu$, $(20)\wedge(9): w=(uts)^2utut$,
$(20)\wedge(12): w=(uts)^2ustsus$, $(20)\wedge(13): w=(uts)^2us(tu)^2$,
$(20)\wedge(14): w=(uts)^2utsust$, $(20)\wedge(17): w=(uts)^2ustsuts$,
$(20)\wedge(18): w=(uts)^2(ust)^2u$, $(20)\wedge(19): w=(uts)^2(uts)^2t$,
$(20)\wedge(20): w=(uts)^2(uts)^2u$, $(20)\wedge(21): w=(uts)^2utsutus$,
$(20)\wedge(22): w=(uts)^2utu(st)^2$, $(20)\wedge(23): w=(uts)^2utustut$,
$(20)\wedge(26): w=(uts)^2ustutsus$, $(20)\wedge(27): w=(uts)^2ustutsut$,
$(20)\wedge(28): w=(uts)^2(utst)^2$, $(20)\wedge(34): w=(uts)^2(ust)^2sut$,
$(20)\wedge(35): w=(uts)^2u(stut)^2$, $(20)\wedge(36): w=(uts)^2utstutsut$,
$(20)\wedge(41): w=(uts)^2us(tstu)^2$, $(20)\wedge(42): w=(uts)^2u(st)^2utsut$,
$(20)\wedge(43): w=(uts)^2ustsutusts$, $(20)\wedge(44): w=(uts)^2ustsutustu$,

$(20) \wedge (45) : w = (uts)^2(ust)^2stut,$ $(20) \wedge (46) : w = (uts)^2ustutstust,$
$(20) \wedge (47) : w = (uts)^2uts(tu)^2sts,$ $(20) \wedge (48) : w = (uts)^2uts(tu)^2stu,$
$(20) \wedge (49) : w = (uts)^2utustsutus,$ $(20) \wedge (50) : w = (uts)^2u(tus)^2tst,$
$(20) \wedge (53) : w = (uts)^2u(st)^2utusts,$ $(20) \wedge (54) : w = (uts)^2u(st)^2utustu,$
$(20) \wedge (55) : w = (uts)^2u(stust)^2,$ $(21) \wedge (7) : w = utsutusu,$
$(21) \wedge (12) : w = utsutustsus,$ $(21) \wedge (13) : w = utsutus(tu)^2,$
$(21) \wedge (17) : w = utsutustsuts,$ $(21) \wedge (18) : w = utsut(ust)^2u,$
$(21) \wedge (26) : w = utsutustutsus,$ $(21) \wedge (27) : w = utsutustutsut,$
$(21) \wedge (34) : w = utsut(ust)^2sut,$ $(21) \wedge (35) : w = utsutu(stut)^2,$
$(21) \wedge (41) : w = utsutus(tstu)^2,$ $(21) \wedge (42) : w = utsutu(st)^2utsut,$
$(21) \wedge (43) : w = utsutustsutusts,$ $(21) \wedge (44) : w = utsutustsutustu,$
$(21) \wedge (45) : w = utsut(ust)^2stut,$ $(21) \wedge (46) : w = utsutustutstust,$
$(21) \wedge (53) : w = utsutu(st)^2utusts,$ $(21) \wedge (54) : w = utsutu(st)^2utustu,$
$(21) \wedge (55) : w = utsutu(stust)^2,$ $(22) \wedge (5) : w = utustst^2,$
$(22) \wedge (8) : w = utusts(ts)^2,$ $(22) \wedge (10) : w = utuststsusts,$
$(22) \wedge (11) : w = utuststsustu,$ $(22) \wedge (15) : w = utuststst ustu,$
$(22) \wedge (16) : w = utustst(stu)^2,$ $(22) \wedge (24) : w = utuststustutsu,$
$(22) \wedge (25) : w = utusts(tuts)^2,$ $(22) \wedge (29) : w = utusts(tstu)^2t,$
$(22) \wedge (30) : w = utuststsutstust,$ $(22) \wedge (31) : w = utuststsutstuts,$
$(22) \wedge (32) : w = utusts(tus)^2tsu,$ $(22) \wedge (33) : w = utuststustutstu,$
$(22) \wedge (37) : w = utusts(tstu)^2st,$ $(22) \wedge (38) : w = utuststsu(tus)^2t,$
$(22) \wedge (39) : w = utuststu(st)^2utsu,$ $(22) \wedge (40) : w = utusts(tus)^2tstu,$
$(22) \wedge (51) : w = utuststst u(tus)^2t,$ $(22) \wedge (52) : w = utuststsuts(tu)^2st,$
$(23) \wedge (5) : w = utustut^2,$ $(23) \wedge (8) : w = utustu(ts)^2,$
$(23) \wedge (10) : w = utustutsusts,$ $(23) \wedge (11) : w = utustutsustu,$
$(23) \wedge (15) : w = utustutstustu,$ $(23) \wedge (16) : w = utustut(stu)^2,$
$(23) \wedge (24) : w = utustutustutsu,$ $(23) \wedge (25) : w = utustu(tuts)^2,$
$(23) \wedge (29) : w = utustu(tstu)^2t,$ $(23) \wedge (30) : w = utustutsutstust,$
$(23) \wedge (31) : w = utustutsutstuts,$ $(23) \wedge (32) : w = utustu(tus)^2tsu,$
$(23) \wedge (33) : w = utustutustutstu,$ $(23) \wedge (37) : w = utustu(tstu)^2st,$
$(23) \wedge (38) : w = utustutsu(tus)^2t,$ $(23) \wedge (39) : w = utustutu(st)^2utsu,$
$(23) \wedge (40) : w = utustu(tus)^2tstu,$ $(23) \wedge (51) : w = utustutstu(tus)^2t,$
$(23) \wedge (52) : w = utustutsuts(tu)^2st,$ $(24) \wedge (6) : w = tustutsu^2,$

$(24) \wedge (7) : w = tustutsusu,$ $(24) \wedge (9) : w = tustutsutut,$

$(24) \wedge (12) : w = tustutsustsus,$ $(24) \wedge (13) : w = tustutsus(tu)^2,$

$(24) \wedge (14) : w = tustutsutsust,$ $(24) \wedge (17) : w = tustutsustsuts,$

$(24) \wedge (18) : w = tustuts(ust)^2u,$ $(24) \wedge (19) : w = tustuts(uts)^2t,$

$(24) \wedge (20) : w = tustuts(uts)^2u,$ $(24) \wedge (21) : w = tustutsutsutus,$

$(24) \wedge (22) : w = tustutsutu(st)^2,$ $(24) \wedge (23) : w = tustutsutustut,$

$(24) \wedge (26) : w = tustutsustutsus,$ $(24) \wedge (27) : w = tustutsustutsut,$

$(24) \wedge (28) : w = tustuts(utst)^2,$ $(24) \wedge (34) : w = tustuts(ust)^2sut,$

$(24) \wedge (35) : w = tustutsu(stut)^2,$ $(24) \wedge (36) : w = tustutsutstutsut,$

$(24) \wedge (41) : w = tustutsus(tstu)^2,$ $(24) \wedge (42) : w = tustutsu(st)^2utsut,$

$(24) \wedge (43) : w = tustutsustsutusts,$ $(24) \wedge (44) : w = tustutsustsutustu,$

$(24) \wedge (45) : w = tustuts(ust)^2stut,$ $(24) \wedge (46) : w = tustutsustutstust,$

$(24) \wedge (47) : w = tustutsuts(tu)^2sts,$ $(24) \wedge (48) : w = tustutsuts(tu)^2stu,$

$(24) \wedge (49) : w = tustutsutustsutus,$ $(24) \wedge (50) : w = tustutsu(tus)^2tst,$

$(24) \wedge (53) : w = tustutsu(st)^2utusts,$ $(24) \wedge (54) : w = tustutsu(st)^2utustu,$

$(24) \wedge (55) : w = tustutsu(stust)^2,$ $(25) \wedge (8) : w = tutstu(ts)^2,$

$(25) \wedge (15) : w = tutstutstustu,$ $(25) \wedge (16) : w = tutstut(stu)^2,$

$(25) \wedge (29) : w = tutstu(tstu)^2t,$ $(25) \wedge (30) : w = tutstutsutstust,$

$(25) \wedge (31) : w = tutstutsutstuts,$ $(25) \wedge (37) : w = tutstu(tstu)^2st,$

$(25) \wedge (38) : w = tutstutsu(tus)^2t,$ $(25) \wedge (51) : w = tutstutstu(tus)^2t,$

$(25) \wedge (52) : w = tutstutsuts(tu)^2st,$ $(26) \wedge (7) : w = ustutsusu,$

$(26) \wedge (12) : w = ustutsustsus,$ $(26) \wedge (13) : w = ustutsus(tu)^2,$

$(26) \wedge (17) : w = ustutsustsuts,$ $(26) \wedge (18) : w = ustuts(ust)^2u,$

$(26) \wedge (26) : w = ustutsustutsus,$ $(26) \wedge (27) : w = ustutsustutsut,$

$(26) \wedge (34) : w = ustuts(ust)^2sut,$ $(26) \wedge (35) : w = ustutsu(stut)^2,$

$(26) \wedge (41) : w = ustutsus(tstu)^2,$ $(26) \wedge (42) : w = ustutsu(st)^2utsut,$

$(26) \wedge (43) : w = ustutsustsutusts,$ $(26) \wedge (44) : w = ustutsustsutustu,$

$(26) \wedge (45) : w = ustuts(ust)^2stut,$ $(26) \wedge (46) : w = ustutsustutstust,$

$(26) \wedge (53) : w = ustutsu(st)^2utusts,$ $(26) \wedge (54) : w = ustutsu(st)^2utustu,$

$(26) \wedge (55) : w = ustutsu(stust)^2,$ $(27) \wedge (5) : w = ustutsut^2,$

$(27) \wedge (8) : w = ustutsu(ts)^2,$ $(27) \wedge (10) : w = ustutsutsusts,$

$(27) \wedge (11) : w = ustutsutsustu,$ $(27) \wedge (15) : w = ustutsutstustu,$

$(27) \wedge (16) : w = ustutsut(stu)^2,$ $(27) \wedge (24) : w = ustutsutustutsu,$

$(27) \wedge (25) : w = ustutsu(tuts)^2,$ $(27) \wedge (29) : w = ustutsu(tstu)^2t,$
$(27) \wedge (30) : w = ustutsutsutstust,$ $(27) \wedge (31) : w = ustutsutsutstuts,$
$(27) \wedge (32) : w = ustutsu(tus)^2tsu,$ $(27) \wedge (33) : w = ustutsutustutstu,$
$(27) \wedge (37) : w = ustutsu(tstu)^2st,$ $(27) \wedge (38) : w = ustutsutsu(tus)^2t,$
$(27) \wedge (39) : w = ustutsutu(st)^2utsu,$ $(27) \wedge (40) : w = ustutsu(tus)^2tstu$
$(27) \wedge (51) : w = ustutsutstu(tus)^2t,$ $(27) \wedge (52) : w = ustutsutsuts(tu)^2st,$
$(28) \wedge (5) : w = utstutst^2,$ $(28) \wedge (8) : w = utstuts(ts)^2,$
$(28) \wedge (10) : w = utstutstsusts,$ $(28) \wedge (11) : w = utstutstsustu,$
$(28) \wedge (15) : w = utstutststustu,$ $(28) \wedge (16) : w = utstutst(stu)^2,$
$(28) \wedge (24) : w = utstutstustutsu,$ $(28) \wedge (25) : w = utstuts(tuts)^2,$
$(28) \wedge (29) : w = utstuts(tstu)^2t,$ $(28) \wedge (30) : w = utstutstsutstust,$
$(28) \wedge (31) : w = utstutstsutstuts,$ $(28) \wedge (32) : w = utstuts(tus)^2tsu,$
$(28) \wedge (33) : w = utstutstustutstu,$ $(28) \wedge (37) : w = utstuts(tstu)^2st,$
$(28) \wedge (38) : w = utstutstsu(tus)^2t,$ $(28) \wedge (39) : w = utstutstu(st)^2utsu,$
$(28) \wedge (40) : w = utstuts(tus)^2tstu,$ $(28) \wedge (51) : w = utstutststu(tus)^2t,$
$(28) \wedge (52) : w = utstutstsuts(tu)^2st,$ $(29) \wedge (5) : w = (tstu)^2t^2,$
$(29) \wedge (8) : w = (tstu)^2(ts)^2,$ $(29) \wedge (10) : w = (tstu)^2tsusts,$
$(29) \wedge (11) : w = (tstu)^2tsustu,$ $(29) \wedge (15) : w = (tstu)^2tstustu,$
$(29) \wedge (16) : w = (tstu)^2t(stu)^2,$ $(29) \wedge (24) : w = (tstu)^2tustutsu,$
$(29) \wedge (25) : w = (tstu)^2(tuts)^2,$ $(29) \wedge (29) : w = (tstu)^2(tstu)^2t,$
$(29) \wedge (30) : w = (tstu)^2tsutstust,$ $(29) \wedge (31) : w = (tstu)^2tsutstuts,$
$(29) \wedge (32) : w = (tstu)^2(tus)^2tsu,$ $(29) \wedge (33) : w = (tstu)^2tustutstu,$
$(29) \wedge (37) : w = (tstu)^2(tstu)^2st,$ $(29) \wedge (38) : w = (tstu)^2tsu(tus)^2t,$
$(29) \wedge (39) : w = (tstu)^2tu(st)^2utsu,$ $(29) \wedge (40) : w = (tstu)^2(tus)^2tstu,$
$(29) \wedge (51) : w = (tstu)^2tstu(tus)^2t,$ $(29) \wedge (52) : w = (tstu)^2tsuts(tu)^2st,$
$(30) \wedge (5) : w = tsutstust^2,$ $(30) \wedge (8) : w = tsutstus(ts)^2,$
$(30) \wedge (10) : w = tsutstustsusts,$ $(30) \wedge (11) : w = tsutstustsustu,$
$(30) \wedge (15) : w = tsutstuststustu,$ $(30) \wedge (16) : w = tsutstust(stu)^2,$
$(30) \wedge (24) : w = tsutstustustutsu,$ $(30) \wedge (25) : w = tsutstus(tuts)^2,$
$(30) \wedge (29) : w = tsutstus(tstu)^2t,$ $(30) \wedge (30) : w = tsutstustsutstust,$
$(30) \wedge (31) : w = tsutstustsutstuts,$ $(30) \wedge (32) : w = tsutstus(tus)^2tsu,$
$(30) \wedge (33) : w = tsutstustustutstu,$ $(30) \wedge (37) : w = tsutstus(tstu)^2st,$
$(30) \wedge (38) : w = tsutstustsu(tus)^2t,$ $(30) \wedge (39) : w = tsutstustu(st)^2utsu,$

$(30) \wedge (40) : w = tsutstus(tus)^2tstu,$ $(30) \wedge (51) : w = tsutstuststu(tus)^2t,$

$(30) \wedge (52) : w = tsutstustsuts(tu)^2st,$ $(31) \wedge (8) : w = tsutstu(ts)^2,$

$(31) \wedge (15) : w = tsutstutstustu,$ $(31) \wedge (16) : w = tsutstut(stu)^2,$

$(31) \wedge (29) : w = tsutstu(tstu)^2t,$ $(31) \wedge (30) : w = tsutstutsutstust,$

$(31) \wedge (31) : w = tsutstutsutstuts,$ $(31) \wedge (37) : w = tsutstu(tstu)^2st,$

$(31) \wedge (38) : w = tsutstutsu(tus)^2t,$ $(31) \wedge (51) : w = tsutstutstu(tus)^2t,$

$(31) \wedge (52) : w = tsutstutsuts(tu)^2st,$ $(32) \wedge (6) : w = (tus)^2tsu^2,$

$(32) \wedge (7) : w = (tus)^2tsusu,$ $(32) \wedge (9) : w = (tus)^2tsutut,$

$(32) \wedge (12) : w = (tus)^2tsustsus,$ $(32) \wedge (13) : w = (tus)^2tsus(tu)^2,$

$(32) \wedge (14) : w = (tus)^2tsutsust,$ $(32) \wedge (17) : w = (tus)^2tsustsuts,$

$(32) \wedge (18) : w = (tus)^2ts(ust)^2u,$ $(32) \wedge (19) : w = (tus)^2ts(uts)^2t,$

$(32) \wedge (20) : w = (tus)^2ts(uts)^2u,$ $(32) \wedge (21) : w = (tus)^2tsutsutus,$

$(32) \wedge (22) : w = (tus)^2tsutu(st)^2,$ $(32) \wedge (23) : w = (tus)^2tsutustut,$

$(32) \wedge (26) : w = (tus)^2tsustutsus,$ $(32) \wedge (27) : w = (tus)^2tsustutsut,$

$(32) \wedge (28) : w = (tus)^2ts(utst)^2,$ $(32) \wedge (34) : w = (tus)^2ts(ust)^2sut,$

$(32) \wedge (35) : w = (tus)^2tsu(stut)^2,$ $(32) \wedge (36) : w = (tus)^2tsutstutsut,$

$(32) \wedge (41) : w = (tus)^2tsus(tstu)^2,$ $(32) \wedge (42) : w = (tus)^2tsu(st)^2utsut,$

$(32) \wedge (43) : w = (tus)^2tsustsutusts,$ $(32) \wedge (44) : w = (tus)^2tsustsutustu,$

$(32) \wedge (45) : w = (tus)^2ts(ust)^2stut,$ $(32) \wedge (46) : w = (tus)^2tsustutstust,$

$(32) \wedge (47) : w = (tus)^2tsuts(tu)^2sts,$ $(32) \wedge (48) : w = (tus)^2tsuts(tu)^2stu,$

$(32) \wedge (49) : w = (tus)^2tsutustsutus,$ $(32) \wedge (50) : w = (tus)^2tsu(tus)^2tst,$

$(32) \wedge (53) : w = (tus)^2tsu(st)^2utusts,$ $(32) \wedge (54) : w = (tus)^2tsu(st)^2utustu,$

$(32) \wedge (55) : w = (tus)^2tsu(stust)^2,$ $(33) \wedge (6) : w = tustutstu^2,$

$(33) \wedge (7) : w = tustutstusu,$ $(33) \wedge (9) : w = tustutstutut,$

$(33) \wedge (12) : w = tustutstustsus,$ $(33) \wedge (13) : w = tustutstus(tu)^2,$

$(33) \wedge (14) : w = tustutstutsust,$ $(33) \wedge (17) : w = tustutstustsuts,$

$(33) \wedge (18) : w = tustutst(ust)^2u,$ $(33) \wedge (19) : w = tustutst(uts)^2t,$

$(33) \wedge (20) : w = tustutst(uts)^2u,$ $(33) \wedge (21) : w = tustutstutsutus,$

$(33) \wedge (22) : w = tustutstutu(st)^2,$ $(33) \wedge (23) : w = tustutstutustut,$

$(33) \wedge (26) : w = tustutstustutsus,$ $(33) \wedge (27) : w = tustutstustutsut,$

$(33) \wedge (28) : w = tustutst(utst)^2,$ $(33) \wedge (34) : w = tustutst(ust)^2sut,$

$(33) \wedge (35) : w = tustutstu(stut)^2,$ $(33) \wedge (36) : w = tustutstutstutsut,$

$(33) \wedge (41) : w = tustutstus(tstu)^2,$ $(33) \wedge (42) : w = tustutstu(st)^2utsut,$

$(33) \wedge (43) : w = tustutstustsutusts,$ $(33) \wedge (44) : w = tustutstustsutustu,$
$(33) \wedge (45) : w = tustutst(ust)^2stut,$ $(33) \wedge (46) : w = tustutstustutstust,$
$(33) \wedge (47) : w = tustutstuts(tu)^2sts,$ $(33) \wedge (46) : w = tustutstustutstu,$
$(33) \wedge (48) : w = tustutstuts(tu)^2stu,$ $(33) \wedge (49) : w = tustutstutustsutus,$
$(33) \wedge (50) : w = tustutstu(tus)^2tst,$ $(33) \wedge (53) : w = tustutstu(st)^2utusts,$
$(33) \wedge (54) : w = tustutstu(st)^2utustu,$ $(33) \wedge (55) : w = tustutstu(stust)^2,$
$(34) \wedge (5) : w = (ust)^2sut^2,$ $(34) \wedge (8) : w = (ust)^2su(ts)^2,$
$(34) \wedge (10) : w = (ust)^2sutsusts,$ $(34) \wedge (11) : w = (ust)^2sutsustu,$
$(34) \wedge (15) : w = (ust)^2sutstustu,$ $(34) \wedge (16) : w = (ust)^2sut(stu)^2,$
$(34) \wedge (24) : w = (ust)^2sutustutsu,$ $(34) \wedge (25) : w = (ust)^2su(tuts)^2,$
$(34) \wedge (29) : w = (ust)^2su(tstu)^2t,$ $(34) \wedge (30) : w = (ust)^2sutsutstust,$
$(34) \wedge (31) : w = (ust)^2sutsutstuts,$ $(34) \wedge (32) : w = (ust)^2su(tus)^2tsu,$
$(34) \wedge (33) : w = (ust)^2sutustutstu,$ $(34) \wedge (37) : w = (ust)^2su(tstu)^2st,$
$(34) \wedge (38) : w = (ust)^2sutsu(tus)^2t,$ $(34) \wedge (39) : w = (ust)^2sutu(st)^2utsu,$
$(34) \wedge (40) : w = (ust)^2su(tus)^2tstu,$ $(34) \wedge (51) : w = (ust)^2sutstu(tus)^2t,$
$(34) \wedge (52) : w = (ust)^2sutsuts(tu)^2st,$ $(35) \wedge (5) : w = ustutstut^2,$
$(35) \wedge (8) : w = ustutstu(ts)^2,$ $(35) \wedge (10) : w = ustutstutsusts,$
$(35) \wedge (11) : w = ustutstutsustu,$ $(35) \wedge (15) : w = ustutstutstustu,$
$(35) \wedge (16) : w = ustutstut(stu)^2,$ $(35) \wedge (24) : w = ustutstutustutsu,$
$(35) \wedge (25) : w = ustutstu(tuts)^2,$ $(35) \wedge (29) : w = ustutstu(tstu)^2t,$
$(35) \wedge (30) : w = ustutstutsutstust,$ $(35) \wedge (31) : w = ustutstutsutstuts,$
$(35) \wedge (32) : w = ustutstu(tus)^2tsu,$ $(35) \wedge (33) : w = ustutstutustutstu,$
$(35) \wedge (37) : w = ustutstu(tstu)^2st,$ $(35) \wedge (38) : w = ustutstutsu(tus)^2t,$
$(35) \wedge (39) : w = ustutstutu(st)^2utsu,$ $(35) \wedge (40) : w = ustutstu(tus)^2tstu,$
$(35) \wedge (51) : w = ustutstutstu(tus)^2t,$ $(35) \wedge (52) : w = ustutstutsuts(tu)^2st,$
$(36) \wedge (5) : w = utstutsut^2,$ $(36) \wedge (8) : w = utstutsu(ts)^2,$
$(36) \wedge (10) : w = utstutsutsusts,$ $(36) \wedge (11) : w = utstutsutsustu,$
$(36) \wedge (15) : w = utstutsutstustu,$ $(36) \wedge (16) : w = utstutsut(stu)^2,$
$(36) \wedge (24) : w = utstutsutustutsu,$ $(36) \wedge (25) : w = utstutsu(tuts)^2,$
$(36) \wedge (29) : w = utstutsu(tstu)^2t,$ $(36) \wedge (30) : w = utstutsutsutstust,$
$(36) \wedge (31) : w = utstutsutsutstuts,$ $(36) \wedge (32) : w = utstutsu(tus)^2tsu,$
$(36) \wedge (33) : w = utstutsutustutstu,$ $(36) \wedge (37) : w = utstutsu(tstu)^2st,$
$(36) \wedge (38) : w = utstutsutsu(tus)^2t,$ $(36) \wedge (39) : w = utstutsutu(st)^2utsu,$

$(36) \wedge (40) : w = utstutsu(tus)^2tstu,$ $(36) \wedge (51) : w = utstutsutstu(tus)^2t,$
$(36) \wedge (52) : w = utstutsutsuts(tu)^2st,$ $(37) \wedge (5) : w = (tstu)^2st^2,$
$(37) \wedge (8) : w = (tstu)^2s(ts)^2,$ $(37) \wedge (10) : w = (tstu)^2stsusts,$
$(37) \wedge (11) : w = (tstu)^2stsustu,$ $(37) \wedge (15) : w = (tstu)^2ststustu,$
$(37) \wedge (16) : w = (tstu)^2st(stu)^2,$ $(37) \wedge (24) : w = (tstu)^2stustutsu,$
$(37) \wedge (25) : w = (tstu)^2s(tuts)^2,$ $(37) \wedge (29) : w = (tstu)^2s(tstu)^2t,$
$(37) \wedge (30) : w = (tstu)^2stsutstust,$ $(37) \wedge (31) : w = (tstu)^2stsutstuts,$
$(37) \wedge (32) : w = (tstu)^2s(tus)^2tsu,$ $(37) \wedge (33) : w = (tstu)^2stustutstu,$
$(37) \wedge (37) : w = (tstu)^2s(tstu)^2st,$ $(37) \wedge (38) : w = (tstu)^2stsu(tus)^2t,$
$(37) \wedge (39) : w = (tstu)^2stu(st)^2utsu,$ $(37) \wedge (40) : w = (tstu)^2s(tus)^2tstu,$
$(37) \wedge (51) : w = (tstu)^2ststu(tus)^2t,$ $(37) \wedge (52) : w = (tstu)^2stsuts(tu)^2st,$
$(38) \wedge (5) : w = tsu(tus)^2t^2,$ $(38) \wedge (8) : w = tsu(tus)^2(ts)^2,$
$(38) \wedge (10) : w = tsu(tus)^2tsusts,$ $(38) \wedge (11) : w = tsu(tus)^2tsustu,$
$(38) \wedge (15) : w = tsu(tus)^2tstustu,$ $(38) \wedge (16) : w = tsu(tus)^2t(stu)^2,$
$(38) \wedge (24) : w = tsu(tus)^2tustutsu,$ $(38) \wedge (25) : w = tsu(tus)^2(tuts)^2,$
$(38) \wedge (29) : w = tsu(tus)^2(tstu)^2t,$ $(38) \wedge (30) : w = tsu(tus)^2tsutstust,$
$(38) \wedge (31) : w = tsu(tus)^2tsutstuts,$ $(38) \wedge (32) : w = tsu(tus)^2(tus)^2tsu,$
$(38) \wedge (33) : w = tsu(tus)^2tustutstu,$ $(38) \wedge (37) : w = tsu(tus)^2(tstu)^2st,$
$(38) \wedge (38) : w = tsu(tus)^2tsu(tus)^2t,$ $(38) \wedge (39) : w = tsu(tus)^2tu(st)^2utsu,$
$(38) \wedge (40) : w = tsu(tus)^2(tus)^2tstu,$ $(38) \wedge (51) : w = tsu(tus)^2tstu(tus)^2t,$
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$(39) \wedge (7) : w = tu(st)^2utsusu,$ $(39) \wedge (9) : w = tu(st)^2utsutut,$
$(39) \wedge (12) : w = tu(st)^2utsustsus,$ $(39) \wedge (13) : w = tu(st)^2utsus(tu)^2,$
$(39) \wedge (14) : w = tu(st)^2utsutsust,$ $(39) \wedge (17) : w = tu(st)^2utsustsuts,$
$(39) \wedge (18) : w = tu(st)^2uts(ust)^2u,$ $(39) \wedge (19) : w = tu(st)^2uts(uts)^2t,$
$(39) \wedge (20) : w = tu(st)^2uts(uts)^2u,$ $(39) \wedge (21) : w = tu(st)^2utsutsutus,$
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$(39) \wedge (26) : w = tu(st)^2utsustutsus,$ $(39) \wedge (27) : w = tu(st)^2utsustutsut,$
$(39) \wedge (28) : w = tu(st)^2uts(utst)^2,$ $(39) \wedge (34) : w = tu(st)^2uts(ust)^2sut,$
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$(39) \wedge (45) : w = tu(st)^2uts(ust)^2stut,$ $(39) \wedge (46) : w = tu(st)^2utsustutstust,$

$(39) \wedge (47) : w = tu(st)^2utsuts(tu)^2sts,$ $(39) \wedge (48) : w = tu(st)^2utsuts(tu)^2stu,$
$(39) \wedge (49) : w = tu(st)^2utsutustsutus,$ $(39) \wedge (50) : w = tu(st)^2utsu(tus)^2tst,$
$(39) \wedge (53) : w = tu(st)^2utsu(st)^2utusts,$ $(39) \wedge (54) : w = tu(st)^2utsu(st)^2utustu,$
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$(40) \wedge (7) : w = (tus)^2tstusu,$ $(40) \wedge (9) : w = (tus)^2tstutut,$
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$(40) \wedge (28) : w = (tus)^2tst(utst)^2,$ $(40) \wedge (34) : w = (tus)^2tst(ust)^2sut,$
$(40) \wedge (35) : w = (tus)^2tstu(stut)^2,$ $(40) \wedge (36) : w = (tus)^2tstutstutsut,$
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$(41) \wedge (35) : w = uststutstu(stut)^2,$ $(41) \wedge (36) : w = uststutstutstutsut,$
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$(41) \wedge (45) : w = uststutst(ust)^2stut,$ $(41) \wedge (46) : w = uststutstustutstust,$
$(41) \wedge (47) : w = uststutstuts(tu)^2sts,$ $(41) \wedge (48) : w = uststutstuts(tu)^2stu,$

$(41) \wedge (49) : w = uststutstutustsutus,$ $(41) \wedge (50) : w = uststutstu(tus)^2tst,$
$(41) \wedge (53) : w = uststutstu(st)^2utusts,$ $(41) \wedge (54) : w = uststutstu(st)^2utustu,$
$(41) \wedge (55) : w = uststutstu(stust)^2,$ $(42) \wedge (5) : w = u(st)^2utsut^2,$
$(42) \wedge (8) : w = u(st)^2utsu(ts)^2,$ $(42) \wedge (10) : w = u(st)^2utsutsusts,$
$(42) \wedge (11) : w = u(st)^2utsutsustu,$ $(42) \wedge (15) : w = u(st)^2utsutstustu,$
$(42) \wedge (16) : w = u(st)^2utsut(stu)^2,$ $(42) \wedge (24) : w = u(st)^2utsutustutsu,$
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$(42) \wedge (37) : w = u(st)^2utsu(tstu)^2st,$ $(42) \wedge (38) : w = u(st)^2utsutsu(tus)^2t,$
$(42) \wedge (39) : w = u(st)^2utsutu(st)^2utsu,$ $(42) \wedge (40) : w = u(st)^2utsu(tus)^2tstu,$
$(42) \wedge (51) : w = u(st)^2utsutstu(tus)^2t,$ $(42) \wedge (52) : w = u(st)^2utsutsuts(tu)^2st,$
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$(43) \wedge (16) : w = ustsutust(stu)^2,$ $(43) \wedge (29) : w = ustsutus(tstu)^2t,$
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$(44) \wedge (19) : w = ustsutust(uts)^2t,$ $(44) \wedge (20) : w = ustsutust(uts)^2u,$
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$(44) \wedge (23) : w = ustsutustutustut,$ $(44) \wedge (26) : w = ustsutustustutsus,$
$(44) \wedge (27) : w = ustsutustustutsut,$ $(44) \wedge (28) : w = ustsutust(utst)^2,$
$(44) \wedge (34) : w = ustsutust(ust)^2sut,$ $(44) \wedge (35) : w = ustsutustu(stut)^2,$
$(44) \wedge (36) : w = ustsutustutstutsut,$ $(44) \wedge (41) : w = ustsutustus(tstu)^2,$
$(44) \wedge (42) : w = ustsutustu(st)^2utsut,$ $(44) \wedge (43) : w = ustsutustustsutusts,$
$(44) \wedge (44) : w = ustsutustustsutustu,$ $(44) \wedge (45) : w = ustsutust(ust)^2stut,$
$(44) \wedge (46) : w = ustsutustustutstust,$ $(44) \wedge (47) : w = ustsutustuts(tu)^2sts,$
$(44) \wedge (48) : w = ustsutustuts(tu)^2stu,$ $(44) \wedge (49) : w = ustsutustutustsutus,$
$(44) \wedge (50) : w = ustsutustu(tus)^2tst,$ $(44) \wedge (53) : w = ustsutustu(st)^2utusts,$
$(44) \wedge (54) : w = ustsutustu(st)^2utustu,$ $(44) \wedge (55) : w = ustsutustu(stust)^2,$

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$(45) \wedge (8) : w = (ust)^2 stu(ts)^2$,
$(45) \wedge (10) : w = (ust)^2 stutsusts$,
$(45) \wedge (11) : w = (ust)^2 stutsustu$,
$(45) \wedge (15) : w = (ust)^2 stutstustu$,
$(45) \wedge (16) : w = (ust)^2 stut(stu)^2$,
$(45) \wedge (24) : w = (ust)^2 stutustutsu$,
$(45) \wedge (25) : w = (ust)^2 stu(tuts)^2$,
$(45) \wedge (29) : w = (ust)^2 stu(tstu)^2 t$,
$(45) \wedge (30) : w = (ust)^2 stutsutstust$,
$(45) \wedge (31) : w = (ust)^2 stutsutstuts$,
$(45) \wedge (32) : w = (ust)^2 stu(tus)^2 tsu$,
$(45) \wedge (33) : w = (ust)^2 stutustutstu$,
$(45) \wedge (37) : w = (ust)^2 stu(tstu)^2 st$,
$(45) \wedge (38) : w = (ust)^2 stutsu(tus)^2 t$,
$(45) \wedge (39) : w = (ust)^2 stutu(st)^2 utsu$,
$(45) \wedge (40) : w = (ust)^2 stu(tus)^2 tstu$,
$(45) \wedge (51) : w = (ust)^2 stutstu(tus)^2 t$,
$(45) \wedge (52) : w = (ust)^2 stutsuts(tu)^2 st$,
$(46) \wedge (5) : w = ustutstust^2$,
$(46) \wedge (8) : w = ustutstus(ts)^2$,
$(46) \wedge (10) : w = ustutstustsusts$,
$(46) \wedge (11) : w = ustutstustsustu$,
$(46) \wedge (15) : w = ustutstuststustu$,
$(46) \wedge (16) : w = ustutstust(stu)^2$,
$(46) \wedge (24) : w = ustutstustustutsu$,
$(46) \wedge (25) : w = ustutstus(tuts)^2$,
$(46) \wedge (29) : w = ustutstus(tstu)^2 t$,
$(46) \wedge (30) : w = ustutstustsutstust$,
$(46) \wedge (31) : w = ustutstustsutstuts$,
$(46) \wedge (32) : w = ustutstus(tus)^2 tsu$,
$(46) \wedge (33) : w = ustutstustustutstu$,
$(46) \wedge (37) : w = ustutstus(tstu)^2 st$,
$(46) \wedge (38) : w = ustutstustsu(tus)^2 t$,
$(46) \wedge (39) : w = ustutstustu(st)^2 utsu$,
$(46) \wedge (40) : w = ustutstus(tus)^2 tstu$,
$(46) \wedge (51) : w = ustutstuststu(tus)^2 t$,
$(46) \wedge (52) : w = ustutstustsuts(tu)^2 st$,
$(47) \wedge (8) : w = uts(tu)^2 s(ts)^2$,
$(47) \wedge (15) : w = uts(tu)^2 ststustu$,
$(47) \wedge (16) : w = uts(tu)^2 st(stu)^2$,
$(47) \wedge (29) : w = uts(tu)^2 s(tstu)^2 t$,
$(47) \wedge (30) : w = uts(tu)^2 stsutstust$,
$(47) \wedge (31) : w = uts(tu)^2 stsutstuts$,
$(47) \wedge (37) : w = uts(tu)^2 s(tstu)^2 st$,
$(47) \wedge (38) : w = uts(tu)^2 stsu(tus)^2 t$,
$(47) \wedge (51) : w = uts(tu)^2 ststu(tus)^2 t$,
$(47) \wedge (52) : w = uts(tu)^2 stsuts(tu)^2 st$,
$(48) \wedge (6) : w = uts(tu)^2 stu^2$,
$(48) \wedge (7) : w = uts(tu)^2 stusu$,
$(48) \wedge (9) : w = uts(tu)^2 stutut$,
$(48) \wedge (12) : w = uts(tu)^2 stustsus$,
$(48) \wedge (13) : w = uts(tu)^2 stus(tu)^2$,
$(48) \wedge (14) : w = uts(tu)^2 stutsust$,
$(48) \wedge (17) : w = uts(tu)^2 stustsuts$,
$(48) \wedge (18) : w = uts(tu)^2 st(ust)^2 u$,
$(48) \wedge (19) : w = uts(tu)^2 st(uts)^2 t$,
$(48) \wedge (20) : w = uts(tu)^2 st(uts)^2 u$,
$(48) \wedge (21) : w = uts(tu)^2 stutsutus$,
$(48) \wedge (22) : w = uts(tu)^2 stutu(st)^2$,
$(48) \wedge (23) : w = uts(tu)^2 stutustut$,
$(48) \wedge (26) : w = uts(tu)^2 stustutsus$,
$(48) \wedge (27) : w = uts(tu)^2 stustutsut$,
$(48) \wedge (28) : w = uts(tu)^2 st(utst)^2$,
$(48) \wedge (34) : w = uts(tu)^2 st(ust)^2 sut$,
$(48) \wedge (35) : w = uts(tu)^2 stu(stut)^2$,

$(48) \wedge (36) : w = uts(tu)^2stutstutsut,$ $(48) \wedge (41) : w = uts(tu)^2stus(tstu)^2,$
$(48) \wedge (42) : w = uts(tu)^2stu(st)^2utsut,$ $(48) \wedge (43) : w = uts(tu)^2stustsutusts,$
$(48) \wedge (44) : w = uts(tu)^2stustsutustu,$ $(48) \wedge (45) : w = uts(tu)^2st(ust)^2stut,$
$(48) \wedge (46) : w = uts(tu)^2stustutstust,$ $(48) \wedge (47) : w = uts(tu)^2stuts(tu)^2sts,$
$(48) \wedge (48) : w = uts(tu)^2stuts(tu)^2stu,$ $(48) \wedge (49) : w = uts(tu)^2stutustsutus,$
$(48) \wedge (50) : w = uts(tu)^2stu(tus)^2tst,$ $(48) \wedge (53) : w = uts(tu)^2stu(st)^2utusts,$
$(48) \wedge (54) : w = uts(tu)^2stu(st)^2utustu,$ $(48) \wedge (55) : w = uts(tu)^2stu(stust)^2,$
$(49) \wedge (7) : w = utustsutusu,$ $(49) \wedge (12) : w = utustsutustsus,$
$(49) \wedge (13) : w = utustsutus(tu)^2,$ $(49) \wedge (17) : w = utustsutustsuts,$
$(49) \wedge (18) : w = utustsut(ust)^2u,$ $(49) \wedge (26) : w = utustsutustutsus,$
$(49) \wedge (27) : w = utustsutustutsut,$ $(49) \wedge (34) : w = utustsut(ust)^2sut,$
$(49) \wedge (35) : w = utustsutu(stut)^2,$ $(49) \wedge (41) : w = utustsutus(tstu)^2,$
$(49) \wedge (42) : w = utustsutu(st)^2utsut,$ $(49) \wedge (43) : w = utustsutustsutusts,$
$(49) \wedge (44) : w = utustsutustsutustu,$ $(49) \wedge (45) : w = utustsut(ust)^2stut,$
$(49) \wedge (46) : w = utustsutustutstust,$ $(49) \wedge (53) : w = utustsutu(st)^2utusts,$
$(49) \wedge (54) : w = utustsutu(st)^2utustu,$ $(49) \wedge (55) : w = utustsutu(stust)^2,$
$(50) \wedge (5) : w = u(tus)^2tst^2,$ $(50) \wedge (8) : w = u(tus)^2ts(ts)^2,$
$(50) \wedge (10) : w = u(tus)^2tstsusts,$ $(50) \wedge (11) : w = u(tus)^2tstsustu,$
$(50) \wedge (15) : w = u(tus)^2tststustu,$ $(50) \wedge (16) : w = u(tus)^2tst(stu)^2,$
$(50) \wedge (24) : w = u(tus)^2tstustutsu,$ $(50) \wedge (25) : w = u(tus)^2ts(tuts)^2,$
$(50) \wedge (29) : w = u(tus)^2ts(tstu)^2t,$ $(50) \wedge (30) : w = u(tus)^2tstsutstust,$
$(50) \wedge (31) : w = u(tus)^2tstsutstuts,$ $(50) \wedge (32) : w = u(tus)^2ts(tus)^2tsu,$
$(50) \wedge (33) : w = u(tus)^2tstustutstu,$ $(50) \wedge (37) : w = u(tus)^2ts(tstu)^2st,$
$(50) \wedge (38) : w = u(tus)^2tstsu(tus)^2t,$ $(50) \wedge (39) : w = u(tus)^2tstu(st)^2utsu,$
$(50) \wedge (40) : w = u(tus)^2ts(tus)^2tstu,$ $(50) \wedge (51) : w = u(tus)^2tststu(tus)^2t,$
$(50) \wedge (52) : w = u(tus)^2tstsuts(tu)^2st,$ $(51) \wedge (5) : w = tstu(tus)^2t^2,$
$(51) \wedge (8) : w = tstu(tus)^2(ts)^2,$ $(51) \wedge (10) : w = tstu(tus)^2tsusts,$
$(51) \wedge (11) : w = tstu(tus)^2tsustu,$ $(51) \wedge (15) : w = tstu(tus)^2tstustu,$
$(51) \wedge (16) : w = tstu(tus)^2t(stu)^2,$ $(51) \wedge (24) : w = tstu(tus)^2tustutsu,$
$(51) \wedge (25) : w = tstu(tus)^2(tuts)^2,$ $(51) \wedge (29) : w = tstu(tus)^2(tstu)^2t,$
$(51) \wedge (30) : w = tstu(tus)^2tsutstust,$ $(51) \wedge (31) : w = tstu(tus)^2tsutstuts,$
$(51) \wedge (32) : w = tstu(tus)^2(tus)^2tsu,$ $(51) \wedge (33) : w = tstu(tus)^2tustutstu,$
$(51) \wedge (37) : w = tstu(tus)^2(tstu)^2st,$ $(51) \wedge (38) : w = tstu(tus)^2tsu(tus)^2t,$

$(51) \wedge (39) : w = tstu(tus)^2tu(st)^2utsu$, $(51) \wedge (40) : w = tstu(tus)^2(tus)^2tstu$,
$(51) \wedge (51) : w = tstu(tus)^2tstu(tus)^2t$, $(51) \wedge (52) : w = tstu(tus)^2tsuts(tu)^2st$,
$(52) \wedge (5) : w = tsuts(tu)^2st^2$, $(52) \wedge (8) : w = tsuts(tu)^2s(ts)^2$,
$(52) \wedge (10) : w = tsuts(tu)^2stsusts$, $(52) \wedge (11) : w = tsuts(tu)^2stsustu$,
$(52) \wedge (15) : w = tsuts(tu)^2ststustu$, $(52) \wedge (16) : w = tsuts(tu)^2st(stu)^2$,
$(52) \wedge (24) : w = tsuts(tu)^2stustutsu$, $(52) \wedge (25) : w = tsuts(tu)^2s(tuts)^2$,
$(52) \wedge (29) : w = tsuts(tu)^2s(tstu)^2t$, $(52) \wedge (30) : w = tsuts(tu)^2stsutstust$,
$(52) \wedge (31) : w = tsuts(tu)^2stsutstuts$, $(52) \wedge (32) : w = tsuts(tu)^2s(tus)^2tsu$,
$(52) \wedge (33) : w = tsuts(tu)^2stustutstu$, $(52) \wedge (37) : w = tsuts(tu)^2s(tstu)^2st$,
$(52) \wedge (38) : w = tsuts(tu)^2stsu(tus)^2t$, $(52) \wedge (39) : w = tsuts(tu)^2stu(st)^2utsu$,
$(52) \wedge (40) : w = tsuts(tu)^2s(tus)^2tstu$, $(52) \wedge (51) : w = tsuts(tu)^2ststu(tus)^2t$,
$(52) \wedge (52) : w = tsuts(tu)^2stsuts(tu)^2st$, $(53) \wedge (8) : w = u(st)^2utus(ts)^2$,
$(53) \wedge (15) : w = u(st)^2utuststustu$, $(53) \wedge (16) : w = u(st)^2utust(stu)^2$,
$(53) \wedge (29) : w = u(st)^2utus(tstu)^2t$, $(53) \wedge (30) : w = u(st)^2utustsutstust$,
$(53) \wedge (31) : w = u(st)^2utustsutstuts$, $(53) \wedge (37) : w = u(st)^2utus(tstu)^2st$,
$(53) \wedge (38) : w = u(st)^2utustsu(tus)^2t$, $(53) \wedge (51) : w = u(st)^2utuststu(tus)^2t$,
$(53) \wedge (52) : w = u(st)^2utustsuts(tu)^2st$, $(54) \wedge (6) : w = u(st)^2utustu^2$,
$(54) \wedge (7) : w = u(st)^2utustusu$, $(54) \wedge (9) : w = u(st)^2utustutut$,
$(54) \wedge (12) : w = u(st)^2utustustsus$, $(54) \wedge (13) : w = u(st)^2utustus(tu)^2$,
$(54) \wedge (14) : w = u(st)^2utustutsust$, $(54) \wedge (17) : w = u(st)^2utustustsuts$,
$(54) \wedge (18) : w = u(st)^2utust(ust)^2u$, $(54) \wedge (19) : w = u(st)^2utust(uts)^2t$,
$(54) \wedge (20) : w = u(st)^2utust(uts)^2u$, $(54) \wedge (21) : w = u(st)^2utustutsutus$,
$(54) \wedge (22) : w = u(st)^2utustutu(st)^2$, $(54) \wedge (23) : w = u(st)^2utustutustut$,
$(54) \wedge (26) : w = u(st)^2utustustutsus$, $(54) \wedge (27) : w = u(st)^2utustustutsut$,
$(54) \wedge (28) : w = u(st)^2utust(utst)^2$, $(54) \wedge (34) : w = u(st)^2utust(ust)^2sut$,
$(54) \wedge (35) : w = u(st)^2utustu(stut)^2$, $(54) \wedge (36) : w = u(st)^2utustutstutsut$,
$(54) \wedge (41) : w = u(st)^2utustus(tstu)^2$, $(54) \wedge (42) : w = u(st)^2utustu(st)^2utsut$,
$(54) \wedge (43) : w = u(st)^2utustustsutusts$, $(54) \wedge (44) : w = u(st)^2utustustsutustu$,
$(54) \wedge (45) : w = u(st)^2utust(ust)^2stut$, $(54) \wedge (46) : w = u(st)^2utustustutstust$,
$(54) \wedge (47) : w = u(st)^2utustuts(tu)^2sts$, $(54) \wedge (48) : w = u(st)^2utustuts(tu)^2stu$,
$(54) \wedge (49) : w = u(st)^2utustutustsutus$, $(54) \wedge (50) : w = u(st)^2utustu(tus)^2tst$,
$(54) \wedge (53) : w = u(st)^2utustu(st)^2utusts$, $(54) \wedge (54) : w = u(st)^2utustu(st)^2utustu$,
$(54) \wedge (55) : w = u(st)^2utustu(stust)^2$, $(55) \wedge (5) : w = ustuststust^2$,
$(55) \wedge (8) : w = ustuststus(ts)^2$, $(55) \wedge (10) : w = ustuststustsusts$,
$(55) \wedge (11) : w = ustuststustsustu$, $(55) \wedge (15) : w = ustuststust每stustu$,

$(55) \wedge (16) : w = ustuststust(stu)^2,$ $(55) \wedge (24) : w = ustuststustustutsu,$
$(55) \wedge (25) : w = ustuststus(tuts)^2,$ $(55) \wedge (29) : w = ustuststus(tstu)^2 t,$
$(55) \wedge (30) : w = ustuststustsutstust,$ $(55) \wedge (31) : w = ustuststustsutstuts,$
$(55) \wedge (32) : w = ustuststus(tus)^2 tsu,$ $(55) \wedge (33) : w = ustuststustustutstu,$
$(55) \wedge (37) : w = ustuststus(tstu)^2 st,$ $(55) \wedge (38) : w = ustuststustsu(tus)^2 t,$
$(55) \wedge (39) : w = ustuststustu(st)^2 utsu,$ $(55) \wedge (40) : w = ustuststus(tus)^2 tstu,$
$(55) \wedge (51) : w = ustuststuststu(tus)^2 t,$ $(55) \wedge (52) : w = ustuststustsuts(tu)^2 st.$

All these ambiguities are trivial. Let us show some of them.

$$
\begin{aligned}
(13) \wedge (36) : w &= ustututstutsut,\\
(f,g)_w &= (us(tu)^2 - sustut)tstutsut - ustut(utstutsut - tstutsutu)\\
&= ustu \underbrace{tt}_{\text{by }(5)} stutsutu - sustu \underbrace{tt}_{\text{by }(5)} stutsut\\
&\equiv us \underbrace{tustutsu}_{\text{by }(24)} tu - sus \underbrace{tustutsu}_{\text{by }(24)} t\\
&\equiv u \underbrace{ss}_{\text{by }(4)} utstus \underbrace{tt}_{\text{by }(5)} u - su \underbrace{ss}_{\text{by }(4)} utstus \underbrace{tt}_{\text{by }(5)}\\
&\equiv \underbrace{uu}_{\text{by }(6)} tstusu - s \underbrace{uu}_{\text{by }(6)} tstus \equiv tst \underbrace{usu}_{\text{by }(7)} - ststus\\
&\equiv \underbrace{tsts}_{\text{by }(8)} us - ststus \equiv ststus - ststus \equiv 0.
\end{aligned}
$$

$$
\begin{aligned}
(17) \wedge (29) : w &= ustsu(tstu)^2 t = ustsutstutstut,\\
(f,g)_w &= (ustsuts - stsutst)tutstut - ustsu((tstu)^2 t - utstuts)\\
&= usts \underbrace{uu}_{\text{by }(6)} tstuts - stsuts \underbrace{tt}_{\text{by }(5)} utstut\\
&\equiv us \underbrace{tsts}_{\text{by }(8)} tuts - sts \underbrace{utsutst}_{\text{by }(19)} ut\\
&\equiv u \underbrace{ss}_{\text{by }(4)} ts \underbrace{tt}_{\text{by }(5)} uts - st \underbrace{ss}_{\text{by }(4)} utsutsut \equiv utsuts - st \underbrace{utsutsu}_{\text{by }(20)} t\\
&\equiv utsuts - s \underbrace{tt}_{\text{by }(5)} utsutst \equiv utsuts - s \underbrace{utsutst}_{\text{by }(19)}\\
&\equiv utsuts - \underbrace{ss}_{\text{by }(4)} utsuts\\
&\equiv utsuts - utsuts \equiv 0.
\end{aligned}
$$

$$
\begin{aligned}
(42) \wedge (33) : w &= u(st)^2utsutustutstu,\\
(f,g)_w &= (u(st)^2utsut - tsuts(tu)^2s)ustutstu\\
&\quad - u(st)^2utsu(tustutstu - su(tus)^2t)\\
&= u(st)^2uts\underbrace{usu}_{\text{by (7)}}(tus)^2t - tsutstut\underbrace{usu}_{\text{by (7)}}stutstu\\
&\equiv u(st)^2ut\underbrace{ss}_{\text{by (4)}}us(tus)^2t - tsutstutsu\underbrace{ss}_{\text{by (4)}}tutstu\\
&\equiv \underbrace{u(st)^2utustu}_{\text{by (54)}}stust - \underbrace{tsutstuts}_{\text{by (31)}}\underbrace{utut}_{\text{by (9)}}stu\\
&\equiv stuststu\underbrace{tsts}_{\text{by (8)}}tust - stsutstu\underbrace{tt}_{\text{by (5)}}utustu\\
&\equiv stuststusts\underbrace{tt}_{\text{by (5)}}ust - stsutst\underbrace{uu}_{\text{by (6)}}tustu\\
&\equiv stustst\underbrace{ustsus}_{\text{by (12)}}t - stsuts\underbrace{tt}_{\text{by (5)}}ustu\\
&\equiv stuststst sus\underbrace{tt}_{\text{by (5)}} - sts\underbrace{utsust}_{\text{by (14)}}u\\
&\equiv stus\underbrace{tsts}_{\text{by (8)}}tsus - st\underbrace{ss}_{\text{by (4)}}utsusu\\
&\equiv stu\underbrace{ss}_{\text{by (4)}}ts\underbrace{tt}_{\text{by (5)}}sus - stuts\underbrace{usu}_{\text{by (7)}} \equiv stut\underbrace{ss}_{\text{by (4)}}us - stut\underbrace{ss}_{\text{by (4)}}us\\
&\equiv stutus - stutus \equiv 0.\\
(50) \wedge (10) : w &= u(tus)^2tstsusts,\\
(f,g)_w &= (u(tus)^2tst - su(tus)^2ts)susts - u(tus)^2ts(tsusts - sustsu)\\
&= u(tus)^2t\underbrace{ss}_{\text{by (4)}}ustsu - su(tus)^2t\underbrace{ss}_{\text{by (4)}}usts\\
&\equiv utus\underbrace{tustustsu}_{\text{by (32)}} - sut\underbrace{ustustu}_{\text{by (18)}}sts\\
&\equiv utu\underbrace{ss}_{\text{by (4)}}tutstust - sutstustu\underbrace{tsts}_{\text{by (8)}}\\
&\equiv \underbrace{utut}_{\text{by (9)}}utstust - su\underbrace{tstustu}_{\text{by (16)}}stst
\end{aligned}
$$

$$\equiv tut \underbrace{uu}_{\text{by (6)}} tstust - sustustu \underbrace{ss}_{\text{by (4)}} tst \equiv tu \underbrace{tt}_{\text{by (5)}} stust - s \underbrace{ustustu}_{\text{by (18)}} tst$$

$$\equiv tustust - \underbrace{ss}_{\text{by (4)}} tustu \underbrace{tt}_{\text{by (5)}} st \equiv tustust - tustust \equiv 0.$$

It is seen that there are no inclusion composition of relations (1)–(55). Hence, the proof is completed. □

2.2. *A Gröbner–Shirshov basis for congruence class of complex reflection group G_7*

The monoid presentation of the braid group associated with the congruence class of complex reflection group G_7 is given as follows [22].

$$\mathcal{P}_{G_7} = \langle s, t, u, s^{-1}, t^{-1}, u^{-1}; t^2 = u^3 = s^3 = 1, tus = ust = stu, ss^{-1} = s^{-1}s = 1,$$
$$tt^{-1} = t^{-1}t = 1, uu^{-1} = u^{-1}u = 1\rangle. \quad (2.2)$$

Theorem 2.2. *The congruence class of complex reflection group G_7 has a Gröbner–Shirshov basis with respect to the degree-lexicographic order $t^{-1} > t > u^{-1} > u > s^{-1} > s$ as follows*:

(1) $t^{-1} = t$, (2) $t^2 = 1$, (3) $u^2 = u^{-1}$,
(4) $uu^{-1} = 1$, (5) $s^2 = s^{-1}$, (6) $ss^{-1} = 1$,
(7) $u^{-2} = u$, (8) $u^{-1}u = 1$, (9) $s^{-2} = s$,
(10) $s^{-1}s = 1$, (11) $tus = ust$, (12) $u^{-1}ts = su^{-1}t$,
(13) $tus = stu$, (14) $ts^{-1}u = uts^{-1}$, (15) $tst = usu^{-1}$,
(16) $ts^{-1}t = us^{-1}u^{-1}$, (17) $u^{-1}tu = sts^{-1}$, (18) $ts^{-1}u^{-1} = u^{-1}ts^{-1}$,
(19) $u^{-1}st = stu^{-1}$, (20) $tus^{-1} = s^{-1}tu$, (21) $tut = s^{-1}us$,
(22) $ts^{-1}u^{-1} = s^{-1}u^{-1}t$, (23) $tu^{-1}t = s^{-1}u^{-1}s$, (24) $tuts = s^{-1}us^{-1}$,
(25) $tusu = stu^{-1}$, (26) $tus^{-1}t = sus$, (27) $tus^{-1}u = s^{-1}tu^{-1}$,
(28) $tstu^{-1} = usu$, (29) $tsus = us^{-1}t$, (30) $tsu^{-1}t = s^{-1}u^{-1}s^{-1}$,
(31) $tu^{-1}s^{-1}u^{-1} = s^{-1}ut$, (32) $ts^{-1}tu^{-1} = us^{-1}u$, (33) $ts^{-1}ut = u^{-1}s^{-1}u^{-1}$,
(34) $ts^{-1}us^{-1} = uts$, (35) $ts^{-1}u^{-1}s^{-1} = su^{-1}t$, (36) $utus = stu^{-1}$,
(37) $utst = u^{-1}su^{-1}$, (38) $ts^{-1}ts = utu^{-1}t$, (39) $usut = tu^{-1}s$,
(40) $(tu)^2 = usus^{-1}$, (41) $us^{-1}ts = tsus^{-1}$, (42) $us^{-1}ts^{-1} = tsu$,
(43) $us^{-1}ut = tsu^{-1}s$, (44) $stsu^{-1} = u^{-1}s^{-1}t$, (45) $stu^{-1}t = u^{-1}s$,

(46) $utus^{-1} = stu^{-1}s$, (47) $stu^{-1}s^{-1} = utu$,
(48) $sts^{-1}u = u^{-1}tu^{-1}$, (49) $sts^{-1}u^{-1} = u^{-1}t$,
(50) $(us)^2 = (su)^2$, (51) $(u^{-1}t)^2 = su^{-1}s^{-1}u^{-1}$,
(52) $u^{-1}tu^{-1}s = sut$, (53) $u^{-1}tu^{-1}s^{-1} = suts$,
(54) $u^{-1}sut = utu^{-1}s$, (55) $u^{-1}sus = (st)^2$,
(56) $u^{-1}su^{-1}t = uts$, (57) $u^{-1}s^{-1}tu^{-1} = stsu$,
(58) $u^{-1}s^{-1}ts^{-1} = utsu$, (59) $tutu^{-1} = s^{-1}tst$,
(60) $s^{-1}tsu^{-1} = su^{-1}s^{-1}t$, (61) $s^{-1}tu^{-1}t = su^{-1}s$,
(62) $s^{-1}tu^{-1}s^{-1} = sutu$, (63) $s^{-1}ts^{-1}u = su^{-1}tu^{-1}$,
(64) $s^{-1}ts^{-1}u^{-1} = su^{-1}t$, (65) $s^{-1}utu = tu^{-1}s^{-1}$,
(66) $tu^{-1}s^{-1}u = s^{-1}utu^{-1}$, (67) $tu^{-1}su^{-1} = s^{-1}us^{-1}t$,
(68) $s^{-1}u^{-1}s^{-1}t = tsu^{-1}$, (69) $(ts^{-1})^2 = s^{-1}u^{-1}s^{-1}u$,
(70) $(u^{-1}s^{-1})^2 = (s^{-1}u^{-1})^2$, (71) $(tu)^2t = s^{-1}tu^{-1}s$,
(72) $(tu)^2s^{-1} = (us)^2$, (73) $tutu^{-1}s^{-1} = su^{-1}su$,
(74) $(ts)^2t = us^{-1}tu^{-1}$, (75) $tsutu = (us^{-1})^2$,
(76) $tsu^{-1}su = utu^{-1}s^{-1}$, (77) $tsu^{-1}s^{-1}u^{-1} = s^{-1}uts$,
(78) $tu^{-1}su^{-1}s^{-1} = s^{-1}tsu$, (79) $ts^{-1}tsu = u^{-1}su^{-1}s^{-1}$,
(80) $(ts^{-1})^2t = utsu^{-1}$, (81) $(ut)^2u = u^{-1}sus^{-1}$,
(82) $tsus^{-1}t = (ut)^2u^{-1}$, (83) $utsus^{-1} = u^{-1}s^{-1}ts$,
(84) $utsu^{-1}s = u^{-1}s^{-1}ut$, (85) $tsu^{-1}s^{-1}u = utsu^{-1}s^{-1}$,
(86) $t(su^{-1})^2 = utu^{-1}s^{-1}u$, (87) $tu^{-1}sus^{-1} = us^{-1}tu^{-1}s$,
(88) $tsutu^{-1} = (us^{-1})^2u$, (89) $(st)^2s = u^{-1}sus^{-1}$,
(90) $(st)^2s^{-1} = u^{-1}su$, (91) $stsut = (u^{-1}s)^2$,
(92) $utu^{-1}su = stsus^{-1}$, (93) $sts^{-1}ts = u^{-1}s^{-1}us^{-1}$,
(94) $s(ts^{-1})^2 = u^{-1}s^{-1}u$, (95) $sutus^{-1} = s^{-1}tu^{-1}s$,
(96) $sutsu^{-1} = u^{-1}s^{-1}ut$, (97) $tu^{-1}sus^{-1} = sus^{-1}tu^{-1}$,
(98) $tutu^{-1}s = su^{-1}sus^{-1}$, (99) $tsu^{-1}s^{-1}u = su^{-1}s^{-1}ut$,
(100) $tu^{-1}s^{-1}ut = su^{-1}s^{-1}us^{-1}$, (101) $s(u^{-1}s^{-1})^2 = u^{-1}s^{-1}u^{-1}$,
(102) $tu^{-1}sus^{-1} = u^{-1}sus^{-1}t$, (103) $tsus^{-1}t = u^{-1}sus^{-1}u$,
(104) $(u^{-1}s)^2u = s(us^{-1})^2$, (105) $(u^{-1}s)^2u^{-1} = utu^{-1}s^{-1}t$,
(106) $tsu^{-1}s^{-1}t = u^{-1}su^{-1}s^{-1}u$, (107) $u^{-1}su^{-1}s^{-1}u^{-1} = ts^{-1}ts$,
(108) $u^{-1}s^{-1}tsu = sutu^{-1}s^{-1}$, (109) $tsu^{-1}s^{-1}u = u^{-1}s^{-1}uts$,

(110) $u^{-1}(s^{-1}u)^2 = (su^{-1})^2s^{-1}$, (111) $s^{-1}tsut = (su^{-1})^2s$,

(112) $t(u^{-1}s)^2 = s^{-1}tsus^{-1}$, (113) $tu^{-1}sus^{-1} = s^{-1}tu^{-1}su$,

(114) $s^{-1}tu^{-1}su^{-1} = sus^{-1}t$, (115) $tu^{-1}s^{-1}ut = (s^{-1}t)^2s$,

(116) $(s^{-1}t)^2s^{-1} = su^{-1}s^{-1}u$, (117) $s^{-1}utsu = tsu^{-1}s^{-1}$,

(118) $tsu^{-1}s^{-1}u = s^{-1}utsu^{-1}$, (119) $tu^{-1}s^{-1}ts = (s^{-1}u)^2s^{-1}$,

(120) $tutu^{-1}su^{-1} = s(ut)^2$, (121) $tsutu^{-1}s = utu^{-1}s^{-1}t$,

(122) $t(su^{-1})^2s = u^{-1}su^{-1}s^{-1}t$, (123) $tsu^{-1}s^{-1}ts = utsut$,

(124) $(tsu)^2 = (uts)^2$, (125) $utsutu^{-1} = (su^{-1})^2s^{-1}$,

(126) $utu^{-1}sus^{-1} = sus^{-1}t$, (127) $ts(us^{-1})^2 = ut(u^{-1}s)^2$,

(128) $tsutu^{-1}s^{-1} = utu^{-1}s^{-1}ts$, (129) $(tsu)^2 = (sut)^2$,

(130) $sutu^{-1}su = t(u^{-1}s)^2$, (131) $ts(us^{-1})^2 = sutu^{-1}su^{-1}$,

(132) $tsutu^{-1}s^{-1} = sutu^{-1}s^{-1}t$, (133) $t(su^{-1})^2s^{-1} = (su^{-1})^2s^{-1}$,

(134) $t(su^{-1})^2s^{-1} = u^{-1}su^{-1}s^{-1}ts$.

Proof. Firstly, we see that all relations given in the presentation (2.2) hold in Gröbner–Shirshov basis given above. Now we show that all compositions among relations (1)–(134) are trivial. Just as the proof of Theorem 2.1, the intersection compositions are between the last component of the word and the word which is the same as the first component of the word. Since the intersection compositions are too much, let us give some of them and proofs of this logic. Hence, we have the following ambiguities.

$(2) \wedge (26) : w = t^2us^{-1}t$, $(3) \wedge (39) : w = u^2sut$,

$(11) \wedge (91) : w = tustsut$, $(12) \wedge (49) : w = u^{-1}tsts^{-1}u^{-1}$,

$(15) \wedge (59) : w = tstutu^{-1}$, $(17) \wedge (42) : w = u^{-1}tus^{-1}ts^{-1}$,

$(19) \wedge (2) : w = u^{-1}st^2$, $(20) \wedge (61) : w = tus^{-1}tu^{-1}t$,

$(27) \wedge (36) : w = tus^{-1}utus$, $(30) \wedge (113) : w = tsu^{-1}tu^{-1}sus^{-1}$,

$(43) \wedge (128) : w = us^{-1}utsutu^{-1}s^{-1}$, $(48) \wedge (83) : w = sts^{-1}utsus^{-1}$,

$(62) \wedge (64) : w = s^{-1}tu^{-1}s^{-1}ts^{-1}u^{-1}$, $(66) \wedge (92) : w = tu^{-1}s^{-1}utu^{-1}su$,

$(68) \wedge (122) : w = s^{-1}u^{-1}s^{-1}t(su^{-1})^2s$, $(77) \wedge (110) : w = tsu^{-1}s^{-1}u^{-1}(s^{-1}u)^2$,

$(85) \wedge (41) : w = tsu^{-1}s^{-1}us^{-1}ts$, $(89) \wedge (90) : w = (st)^2(st)^2s^{-1}$,

$(100) \wedge (115) : w = tu^{-1}s^{-1}utu^{-1}s^{-1}ut$, $(112) \wedge (91) : w = t(u^{-1}s)^2tsut = (u^{-1}s)^2$,

$(120) \wedge (53) : w = tutu^{-1}su^{-1}tu^{-1}s^{-1}$, $(130) \wedge (81) : w = sutu^{-1}s(ut)^2u$,

$(4) \wedge (52) : w = uu^{-1}tu^{-1}s$, $(14) \wedge (36) : w = ts^{-1}utus$,

$(18) \wedge (56) : w = ts^{-1}u^{-1}su^{-1}t$, $(26) \wedge (74) : w = tus^{-1}(ts)^2t$,

$(35) \wedge (10) : w = ts^{-1}u^{-1}s^{-1}s,$ $\quad (50) \wedge (91) : w = (us)^2tsut,$

$(67) \wedge (108) : w = tu^{-1}su^{-1}s^{-1}tsu,$ $\quad (80) \wedge (2) : w = (ts^{-1})^2t^2,$

$(93) \wedge (96) : w = sts^{-1}tsutsu^{-1},$ $\quad (123) \wedge (101) : w = tsu^{-1}s^{-1}ts(u^{-1}s^{-1})^2,$

$(134) \wedge (10) : w = t(su^{-1})^2s^{-1}s,$ $\quad (133) \wedge (116) : w = t(su^{-1})^2(s^{-1}t)^2s^{-1}.$

We have written some of these components. All these and other ambiguities are trivial. Let us show two of them as follows:

$$\begin{aligned}
(80) \wedge (2) : w &= (ts^{-1})^2t^2,\\
(f,g)_w &= ((ts^{-1})^2t - utsu^{-1})t - (ts^{-1})^2(t^2-1)\\
&= \underbrace{ts^{-1}t}_{\text{by }(16)}s^{-1} - u\underbrace{tsu^{-1}t}_{\text{by }(30)}\\
&\equiv us^{-1}u^{-1}s^{-1} - us^{-1}u^{-1}s^{-1}\\
&\equiv 0.\\
(26) \wedge (74) : w &= tus^{-1}tstst,\\
(f,g)_w &= (tus^{-1}t - sus)stst - tus^{-1}(tstst - us^{-1}tu^{-1})\\
&= \underbrace{tus^{-1}u}_{\text{by }(27)}s^{-1}tu^{-1} - su\underbrace{ss}_{\text{by }(5)}tst\\
&\equiv \underbrace{s^{-1}tu^{-1}s^{-1}}_{\text{by }(62)}tu^{-1} - sus^{-1}\underbrace{tst}_{\text{by }(15)}\\
&\equiv su\underbrace{tut}_{\text{by }(21)}u^{-1} - sus^{-1}usu^{-1}\\
&\equiv sus^{-1}usu^{-1} - sus^{-1}usu^{-1}\\
&\equiv 0.
\end{aligned}$$

□

We note that in recent years Gröbner–Shirshov basis deserves worldwide attentions due to its remarkable performance. Gröbner–Shirshov basis theory of which starting point is based on the second half of 20th century constitutes a comprehensive study area on computer science as well as computational algebra. In fact, there are other methods that provide the solvability of the word problem which is the main purpose of Gröbner–Shirshov basis. For example, the (string) rewriting system is one of the methods that gives the solvability of the word problem. The best part of the method we use in this paper is that Gröbner–Shirshov basis theory works in larger areas of algebra and mathematics. Gröbner–Shirshov basis theory has applications not only in mathematics but also in theoretical physics.

We will study other congruence classes of complex reflection groups or other symmetrical group types in our future research works.

3. Conclusions

By an accepted package program "idrel" in GAP (Group, Algorithm and Programming) (http://www.gap-system.org/Packages/idrel.html), we checked the correctness of Theorems 2.1 and 2.2. But by using Composition-Diamond Lemma and Theorems 2.1 and 2.2, we can present normal form structure of elements of given groups with a process. We would like to emphasize that no any computer program gives a general normal form structure of elements of an algebra. So it is worth studying to obtain these form structures. As known in literature Gröbner–Shirshov basis theory is the most effective method in this sense.

In this last section, let us give the normal forms of the groups G_{24} and G_7 by using the main results given by Theorems 2.1 and 2.2. To do that, let R_{24} be the set of relations (1)–(55) given in Theorem 2.1 and $C(u)$ be a normal form of a word $u \in G_{24}$. By using the Composition-Diamond Lemma and Theorem 2.1, the normal form for the congruence classes of complex reflection group G_{24} can be given as follows. The normal form for G_{24} will be given as an algorithm.

Corollary 3.1. *$C(u)$ has a form*

$$W^{\alpha_1} W^{\alpha_2} W^{\alpha_3} W^{\alpha_4},$$

where W^{α_4} is an R_{24}-irreducible word and we have the following:

- *if $W^{\alpha_1} = s$ and $W^{\alpha_2} = t$, then W^{α_3} is an R_{24}-irreducible word that does not contain the subword sts,*
- *if $W^{\alpha_1} = s$ and $W^{\alpha_2} = u$, then $W^{\alpha_3} = s$ or $W^{\alpha_3} = t$,*
- *if $W^{\alpha_1} = t$ and $W^{\alpha_2} = s$, then W^{α_3} is an R_{24}-irreducible word that does not contain the subword ts,*
- *if $W^{\alpha_1} = t$ and $W^{\alpha_2} = u$, then W^{α_3} is an R_{24}-irreducible word that does not contain the subword tut,*
- *if $W^{\alpha_1} = u$, then $W^{\alpha_2} = s$ and $W^{\alpha_3} = t$ or $W^{\alpha_2} = t$ and W^{α_3} is an R_{24}-irreducible word that does not contain the subword sts.*

Similarly, let R_7 be the set of relations (1)–(134) given in Theorem 2.2 and $C(u)$ be a normal form of a word $u \in G_7$. By using the Composition-Diamond Lemma and Theorem 2.2, we have the following result.

Corollary 3.2. *$C(u)$ is of the form*

$$sW^{\alpha_1} s^{-1} W^{\alpha_2} u W^{\alpha_3} u^{-1} W^{\alpha_4} t W^{\alpha_5} t^{-1},$$

where

- *W^{α_1} is an R_7-irreducible word that does not contain the generator s and s^{-1},*
- *W^{α_2} is an R_7-irreducible word that does not contain the generator s, s^{-1}, u and u^{-1},*
- *W^{α_3} is an R_7-irreducible word that does not contain the generator u and u^{-1},*

- *W^{α_4} is an R_7-irreducible word that does not contain the generator u, u^{-1}, t and t^{-1},*
- *W^{α_5} is an R_7-irreducible word that does not contain the generator t and t^{-1}.*

By considering Corollaries 3.1 and 3.2, we have the following consequence of our main results.

Corollary 3.3. *The word problem for the congruence classes of complex reflection groups G_{24} and G_7 is solvable.*

References

1. S. I. Adian and V. G. Durnev, Decision problems for groups and semigroups, *Russian Math. Surveys* **55**(2) (2000) 207–296.
2. F. Ateş, E. G. Karpuz, C. Kocapinar and A. S. Çevik, Gröbner–Shirshov bases of some monoids, *Discrete Math.* **311** (2011) 1064–1071.
3. F. Ateş, A. S. Çevik and E. G. Karpuz, Gröbner–Shirshov basis for the singular part of the Brauer semigroup, *Turkish J. Math.* **42** (2018) 1338–1347.
4. G. M. Bergman, The diamond lemma for ring theory, *Adv. Math.* **29** (1978) 178–218.
5. M. Broué, G. Malle and R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, *J. Reine Angew. Math.* **500** (1998) 127–190.
6. B. Buchberger, *An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Ideal*, Ph.D. Thesis, University of Innsbruck (1965).
7. L. A. Bokut and A. Vesnin, Gröbner–Shirshov bases for some Braid groups, *J. Symbol. Comput.* **41** (2006) 357–371.
8. L. A. Bokut, Imbedding into simple associative algebras, *Algebra Logic* **15** (1976) 117–142.
9. L. A. Bokut, Gröbner–Shirshov basis for the Braid group in the Birman-Ko-Lee generators, *J. Algebra* **321** (2009) 361–376.
10. A. M. Cohen, Finite complex reflection groups, *Ann. Scient. Norm. Sup. Ser.* 4^e **9** (1976) 379–436.
11. R. B. Howlett and J. Y. Shi, On regularity of finite reflection groups, *Manuscri. Math.* **102**(3) (2000) 325–333.
12. E. G. Karpuz and A. S. Çevik, Gröbner–Shirshov bases for extended modular, extended Hecke and Picard groups, *Math. Notes* **92**(5) (2012) 636–642.
13. E. G. Karpuz, Gröbner–Shirshov bases of some semigroup constructions, *Algebra Colloq.* **22**(1) (2015) 35–46.
14. E. G. Karpuz, F. Ateş and A. S. Çevik, Gröbner–Shirshov bases of some weyl groups, *Rocky Mountain J. Math.* **45**(4) (2015) 1165–1175.
15. E. G. Karpuz, A. S. Çevik, F. Ateş and J. Koppitz, Gröbner–Shirshov bases and embedding of a semigroup in a group, *Adv. Studies Contemp. Math.* **25**(4) (2015) 537–545.
16. E. G. Karpuz, F. Ateş, N. Urlu, A. S. Çevik and I. N. Cangül, A note on the Gröbner–Shirshov bases over ad-hoc extensions of groups, *Filomat* **30**(4) (2016) 1037–1043.
17. E. G. Karpuz and N. U. Özalan, Word problem for special Braid groups, *Quaest. Mathe.* (2019), doi:10.2989/16073606.2019.1588178.
18. C. Kocapinar, E. G. Karpuz, F. Ateş and A. S. Çevik, Gröbner–Shirshov bases of the generalized Bruck-Reilly $*$-extension, *Algebra Colloq.* **19**(1) (2012) 813–820.
19. G. C. Shephard and J. A. Todd, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954) 274–304.

20. J. Y. Shi, Certain imprimitive reflection groups and their generic versions, *Trans. Amer. Math. Soc.* **354** (2002) 2115–2129.
21. J. Y. Shi, Simple root systems and presentations for certain complex reflection groups, *Commun. Algebra* **6**(33) (2005) 1765–1783.
22. J. Y. Shi, Presentations for finite complex reflection groups, *AMS Ip Stud. Adv. Math.* **39** (2007) 263–276.
23. A. I. Shirshov, Some algorithmic problems for Lie algebras, *Siberian Math. J.* **3** (1962) 292–296.
24. L. Wang, Simple root systems and presentations for the primitive complex reflection groups generated by involutive reflections, Master thesis in ECNU (2003).
25. P. Zeng, Simple root systems and presentations for the primitive complex reflection groups containing reflections of order >2, Master thesis in ECNU (2003).