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Variational Approximate and Mixed-Finite Element Solution for Static Analysis of Laminated Composite Plates

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Abstract. The main objective of the present study is to give a systematic way for the derivation of laminated composite plates by using the mixed type finite element formulation with a functional. The first order shear deformation plate theory is used. Differential field equations of composite plates are derived from virtual displacement principle. These equations were written in operator form then by using the Gâteaux differential method, a new functional which including the dynamic and geometric boundary conditions is obtained after provide potential conditions. Applying mixed-type finite element based on this new functional, a plate element namely FOPLT32 is derived which have 8 degrees of freedoms on per node, total 32 freedoms. The reliability of the derived FOPLT32 plate elements for static analysis is verified, since the results obtained have been shown to agree well with the existing ones.

Introduction

Composite plates are used extensively in aerospace, civil and mechanical engineering for high performance and reliability applications. A number of theories and mathematical models developed for the analysis of plates. There are three well known major theories which are classical plate theory, first order and high order shear deformation plate theories [1]. The classical plate theory namely Kirchhoff plate theory (CPT), which assumes that the straight lines normal to the mid-plane before deformation remain straight and normal to the mid-surface after deformation. This theory has neglected both transverse shear and normal strains. The first order shear deformation theory is including the transverse shear effect with a linear variation of transverse shear strain through the plate thickness. Thus, the use of a shear correction coefficient is necessary. Higher order theories are more valid for shear deformation effects by higher order variations of in-plane displacements and transverse displacements through the thickness. Also satisfy the equilibrium conditions on the top and bottom surfaces of the plate. The shear deformation effects have been represented using parabolic, exponential, trigonometric and hyperbolic functions through the thickness of the plate [2]. The finite element method (FEM) is the most suitable choice for structural analysis, because of its versatility in handling complex geometries and boundary conditions. The Mixed-type FE model is far more efficient due to variables can be chosen independently, the forces and moments can be calculated with less number of elements but more sensitive. In the mixed-type FEM, having field equations one needs a method to obtain the functional. The Gâteaux differential (GD) successfully has employed to construct the functional for various problems which are beam, plate and shell [3-5]. The advantages of the method were summarized by Akoz and Ozutok [3].

In the present study, mixed-type FE formulation is presented for the static analysis of symmetrical laminated composite plate basis on FOPLT. The virtual displacement principle was applied to obtain the field equations. These equations were written in operator form then by using the Gâteaux differential method, the functional which including the dynamic and geometric boundary conditions is obtained after provide potential conditions. The mixed-type finite element FOPLT32 developed based on this function which has four nodes with eight degrees of freedom per node. The displacements, rotations, internal forces, moments of composite plates are calculated independently. A computer program is developed in FORTRAN to carry out the analyses.

Formulation

Consider a rectangular laminated plate of length a, width b and total thickness h with orthotropic layers as shown in Fig. 1. The plate is located in Cartesian coordinate system (x-y-z).

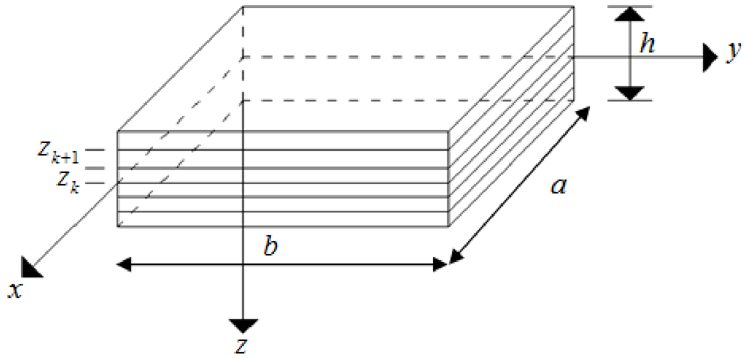

Fig. 1. The geometry of laminated composite plate

Upon the effects of transverse shear deformation and rotary inertia for different plate theories, general displacement fields of laminated composite plates can be defined as Eq. 1.

$$u(x,y,z)=z\phi_x;\quad v(x,y,z)=z\phi_y;\quad w(x,y,z)=w, \tag{1}$$

where the in-plane displacement and transverse displacement components are denoted as u, v and w, respectively. The rotations about y and x axis represented as ϕ_x and ϕ_y. And also "$(...)_{,x}$" and "$(...)_{,y}$" are the partial derivatives with respect to x and y axis, respectively.

The strains associated with the displacement field are written as

$$\begin{aligned}&\varepsilon_x=\frac{\partial u}{\partial x}=z\phi_{x,x};\quad \varepsilon_y=\frac{\partial v}{\partial y}=z\phi_{y,y};\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=z\phi_{x,y}+z\phi_{y,x}\\ &\gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\phi_x+w_{,x};\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\phi_y+w_{,y}\end{aligned} \tag{2}$$

The principle of virtual displacements states that if the plate is in equilibrium we must have the following relation:

$$\begin{aligned}&\delta U=\int_A\int_{-h/2}^{h/2}\left[\left\{\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}+\tau_{xz}\delta\gamma_{xz}+\tau_{yz}\delta\gamma_{yz}\right\}dz\right]dx\,dy\\ &\delta U=\int_A\left\{K\delta\phi_{x,x}+M\delta\phi_{y,y}+T(\delta\phi_{x,y}+\delta\phi_{y,x})+S(\delta\phi_x+\delta w_{,x})+Q(\delta\phi_y+\delta w_{,y})\right\}dxdy,\\ &\delta V=-\int_A q\delta w dxdy\end{aligned} \tag{3}$$

where K, M and T are moments and S and Q are shear forces are defined by

$$\left[K;M;T\right]=\int_{-h/2}^{h/2}[\sigma_x;\sigma_y;\tau_{xy}]\,zdz;\quad \left[S,Q\right]=\int_{-h/2}^{h/2}[\tau_{xz};\tau_{yz}]\,dz. \tag{4}$$

The linear constitutive relations for the kth orthotropic lamina are given in the following equation:

$$\{\sigma\}^{(k)}=\left[Q_{ij}\right]^{(k)}\{\varepsilon\}^{(k)}, \tag{5}$$

where $Q_{ij}^{(k)}$ is the plane stress-reduced stiffness of the kth layer. Based on FOPLT, the field equations of the laminated composite plates can be expressed as

$$
\begin{aligned}
&-S_{,x}-Q_{,y}-q=0\\
&-K_{,x}-T_{,y}+S=0\\
&-M_{,y}-T_{,x}+Q=0\\
&-K+D_{11}\phi_{x,x}+D_{12}\phi_{y,y}=0\\
&-M+D_{12}\phi_{x,x}+D_{22}\phi_{y,y}=0\\
&-T+D_{66}\phi_{x,y}+D_{66}\phi_{y,x}=0\\
&-S+k_s A_{55}\phi_x+k_s A_{55}w_{,x}=0\\
&-Q+k_s A_{44}\phi_y+k_s A_{44}w_{,y}=0
\end{aligned}
\quad . \tag{6}
$$

The laminate rigidities D_{ij} and the transverse shear rigidities A_{ij} appearing in Eq. 7 defined as follows;

$$\left[D_{ij}\right]=\int_{-h/2}^{h/2}\left\{\bar{Q}_{ij}^{(k)}\right\}z^2dz;\quad \left[A_{ij}\right]=k_s\int_{-h/2}^{h/2}\left\{\bar{Q}_{ij}^{(k)}\right\}dz\,. \tag{7}$$

The dynamic boundary conditions and geometric boundary conditions are respectively given as in the following;

$$-\mathbf{R}+\widehat{\mathbf{R}}=0\quad -\mathbf{M}+\widehat{\mathbf{M}}=0;\quad -\mathbf{\Omega}+\widehat{\mathbf{\Omega}}=0\quad -\mathbf{u}+\widehat{\mathbf{u}}=0\,. \tag{8}$$

The explicit form of the boundary conditions will be obtained through the variational processes. In Eq. 8 the quantities can be defined as “force”, “moment”, “rotation” and “deflection” vectors. The field equations including the boundary conditions for composite plates can be written in operator form as

$$\mathbf{Q}=\mathbf{L}\,\mathbf{u}-\mathbf{f}\,. \tag{9}$$

The Gateaux derivative of an operator is defined as

$$d\mathbf{Q}\left(\mathbf{u}\,,\bar{\mathbf{u}}\right)=\left.\frac{\partial\mathbf{Q}\left(\mathbf{u}+\tau\,\bar{\mathbf{u}}\right)}{\partial\tau}\right|_{\tau=0}\,. \tag{10}$$

If the operator $\mathbf{Q}$ is a potential, then the functional which corresponds to the field equations is given

$$I\left(u\right)=\int_0^1\left[\mathbf{Q}\left(s\mathbf{u}\,,\mathbf{u}\right)\right]ds\,. \tag{11}$$

The clear form of the functional corresponding to the field equations are obtained as in the Eq. 12 for FOPLT. Initially, the interpolation function should be selected to derive the mixed FE formulation. The details of the variational procedure and formulation exist in Akoz and Ozutok [3]. All known external and unknown internal quantities expressed with these interpolation functions are inserted into Eq. 12, the element matrices can be obtained as Eq. 13.

$$
\begin{aligned}
I_{[y]FOPLT}=&\left[S,w_{,x}\right]+\left[Q,w_{,y}\right]+\left[S,\phi_x\right]+\left[Q,\phi_y\right]+\left[K,\phi_{x,x}\right]+\left[M,\phi_{y,y}\right]-\frac{D_{22}}{2(D_{11}D_{22}-D_{12}^2)}\left[K,K\right]\\
&-\frac{1}{2}\frac{D_{11}}{D_{11}D_{22}-D_{12}^2}\left[M,M\right]+\frac{D_{12}}{D_{11}D_{22}-D_{12}^2}\left[K,M\right]+\left[T,(\phi_{x,y}+\phi_{y,x})\right]-\frac{1}{2D_{66}}\left[T,T\right]-\frac{1}{2k_sA_{44}}\left[Q,Q\right];\\
&-\frac{1}{2}\frac{1}{k_sA_{55}}\left[S,S\right]-\left[q,w\right]+\left[\left(\widehat{\underset{\sim}{w}}-\underset{\sim}{w}\right),\underset{\sim}{Q}\right]_{\varepsilon}+\left[\left(\widehat{\underset{\sim}{\phi}}-\underset{\sim}{\phi}\right),\underset{\sim}{M}\right]_{\varepsilon}-\left[\widehat{\underset{\sim}{Q}},\underset{\sim}{w}\right]_{\sigma}-\left[\widehat{\underset{\sim}{M}},\underset{\sim}{\phi}\right]_{\sigma}
\end{aligned}
\tag{12}
$$

$$[k_{el}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & [k_2]^T & [k_3]^T \\ & 0 & 0 & [k_2]^T & 0 & [k_3]^T & [k_1]^T & 0 \\ & & 0 & 0 & [k_3]^T & [k_2]^T & 0 & [k_1]^T \\ & & & -\alpha[k_1] & \zeta[k_1] & 0 & 0 & 0 \\ & & & & -\beta[k_1] & 0 & 0 & 0 \\ & sym. & & & & -\Upsilon[k_1] & 0 & 0 \\ & & & & & & -\Phi[k_1] & 0 \\ & & & & & & & -\Delta[k_1] \end{bmatrix} \begin{Bmatrix} [w] \\ [\phi_x] \\ [\phi_y] \\ [K] \\ [M] \\ [T] \\ [S] \\ [Q] \end{Bmatrix} = \begin{bmatrix} q[k_1] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (13)$$

Numerical Results

The numerical results are presented for the laminated plates having a rectangular cross-section. FORTRAN computer program was developed for the analysis. The following non-dimensional terms are used for convenience in some examples, the vertical displacement, in-plane and shear stresses of plates under the uniformly distributed load q:

$$w^* = w\left(\frac{E_2 h^3}{(2b)^4 q_0}\right).100; \quad \sigma^*_{xx}, \sigma^*_{yy}, \tau^*_{xy} = \left[\sigma_{xx}, \sigma_{yy}, \tau_{xy}\right] h^2 / (2b)^2 \quad . \quad (15)$$

$$\tau^*_{xz}, \tau^*_{yz} = \left[\tau_{xz}, \tau_{yz}\right] h / (2b)$$

The mechanical properties of each layer are given in Table 1. All layers have the same thickness. Simply supported plate subjected.

Table 1. Mechanical properties

E_1/E_2	ν_{12}	G_{12}	G_{13}	G_{23}
25	0.25	$0.5E_2$	$0.5E_2$	$0.2E_2$

In this example, a three-layered simply-supported (SSSS) symmetric cross-ply square laminated plate is studied. The non-dimensional deflection and stresses are computed using present method. By varying the side to thickness ratio 10 to 100 (thick to thin), the static analysis is performed and the results are tabulated in Table 2 and Table 3.

Table 2. Non-dimensional maximum displacement (w^*) of SSSS $(0^\circ/90^\circ/0^\circ)$ cross-ply composite plate

Theory	$2a/h$		
	10	20	100
Present	1.0270	0.7612	0.6726
[1]	1.0219	0.7572	0.6697
[6]	1.0291	0.7578	0.6696
[7]	1.0160	0.7548	0.6681

The membrane stress were evaluated at the this locations: $\sigma^*_{xx}(a,b,h/2)$, $\sigma^*_{yy}(a,b,h/4)$ and $\tau^*_{xy}(2a,2b,h/2)$. The transverse shear stresses are calculated using the constitutive equations. The τ^*_{xz} is evaluated at $(2a,0)$ in layers 1 and 3, and τ^*_{yz} is computed at $(0,2b)$ in layers 2. The results of the present method are in excellent agreement with other solutions.

Table 3. Non-dimensional stresses of SSSS $(0^\circ/90^\circ/0^\circ)$ cross-ply composite plate

$2a/h$	Theory	$\sigma^*_{xx}(a,b,h/2)$	$\sigma^*_{yy}(a,b,h/4)$	$\tau^*_{xy}(2a,2b,h/2)$	$\tau^*_{xz}(2a,0)$	$\tau^*_{yz}(0,2b)$
100	Present	0.81106	0.19566	0.0436	0.7926	0.3046
	[1]	0.8072	0.1925	0.0426	0.7744	0.2842
20	Present	0.8030	0.2260	0.0465	0.7883	0.3108
	[1]	0.7983	0.2227	0.0453	0.7697	0.2902
	[8]	0.8125	0.2300	0.0458	…	0.3570
10	Present	0.7767	0.3104	0.0529	0.7737	0.3314
	[1]	0.7719	0.3072	0.0514	0.7548	0.3107
	[8]	0.7660	0.2900	0.0484	…	0.285

Summary

In this study, bending analysis of laminated composite plates was studied by first order shear deformation plate theory. In the analysis, Gâteaux differential method has been used. Using the virtual work principle, the governing differential equations are derived. Boundary conditions terms are constructed and introduced to the functional in a systematic way. The closed form of the element equations FOPLT32 is obtained which eliminate the time-consuming numerical inversion of the element matrix. The Gâteaux differential approach has some advantages; the given field equations are enforced to the functional in a straight forward manner also the boundary conditions for given problem can be constructed easily; it provides the consistency of the field equations.

References

[1] J.N. Reddy, Mechanics of laminated composite plates and shells: theory and analysis: CRC press, 2004.

[2] J. Mantari, A. Oktem, C.G. Soares, A new trigonometric shear deformation theory for isotropic, laminated composite and sandwich plates, Int. J. Solids Struct. 49 (2012) 43-53.

[3] A. Aköz, A. Özütok. A functional for shells of arbitrary geometry and a mixed finite element method for parabolic and circular cylindrical shells, Int. J. Numer. Methods Eng. 47 (2000) 1933-1981.

[4] A. Aköz, N. Uzcan. The new functional for reissner plates and its application, Computers & Structures, 44 (1992) 1139-1144.

[5] A. Özütok, E. Madenci. Free vibration analysis of cross-ply laminated composite beams by mixed finite element formulation, Int. J. Struct. Stab. Dyn. 13 (2013) 1250056.

[6] J. Belinha, L. Dinis. Analysis of plates and laminates using the element-free Galerkin method, Computers & Structures. 84 (2006) 1547-1559.

[7] N. Ateş, G. Tekin, F. Kadıoğlu. Alternative solution for cross-ply laminated composite thick plates, Građevinar. 68 (2016) 451-460.

[8] J. Xiao, D. Gilhooley, R. Batra, J. Gillespie, M. McCarthy. Analysis of thick composite laminates using a higher-order shear and normal deformable plate theory (HOSNDPT) and a meshless method, Composites Part B: Engineering. 39 (2008) 414-427.