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Article *in* Asian-European Journal of Mathematics · June 2019

DOI: 10.1142/S1793557120400082

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Asian-European Journal of Mathematics
Vol. 13, No. 1 (2020) 2040008 (13 pages)

DOI: 10.1142/S1793557120400082

A new semigroup obtained via known ones

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Received May 30, 2019
Accepted June 7, 2019
Published

The goal of this paper is to establish a new class of semigroups based on both Rees matrix and completely 0-simple semigroups. We further present some fundamental properties and finiteness conditions for this new semigroup structure.

Keywords: Rees matrix semigroup; completely 0-simple semigroup; idempotent; Green relations.

AMS Subject Classification: 20M05, 20M17, 20M30

1. Introduction and Preliminaries

Considering a new and more general construction brings several benefits such as unification of already known results in a new structure. For example, in [12], Lipkovski recently presented a new algebraic structure by taking into account an arbitrary commutative ring A with the identity and the mapping $\Phi : A^2 \to A^2$ defined by the Vieta formulas $(x, y) \mapsto (u, v) = (x + y, xy)$, and also he studied the directed graph defined by the Vieta mapping Φ. In the light of a similar approximation for semigroup theory except for the graph case, we will obtain a new semigroup structure which provides a common generalization of the Rees matrix semigroup and completely 0-simple semigroup. Then we will study some fundamental semigroup properties over it. In the literature, there are so many important constructions, for instance, direct, semidirect, free, Mal'cev and Zappa-Szep products which have already used on semigroups that provide tools to decompose on related algebraic structure (see, for example, [4–6, 8, 9]).

It is well known that Rees matrix semigroups were firstly introduced by Rees in [16]. Those construct a special class of semigroups in the meaning of their usage to classify certain classes of (simple) semigroups. Also, these special semigroups became one of the most important semigroup construction with numerous applications, especially in the study of completely 0-simple semigroups (see, for example [10]). The investigation of the Rees matrix semigroup that have largest weights in new constructions based on semigroups has been motivated by many studies, where classification relying on semigroups plays valuable roles. Basically, the class of these semigroups is an important technique for building new semigroups structures out of old ones.

Although the Rees matrix semigroup was defined over groups, it has taken so much interests in semigroups (cf. [2, 3, 11, 15]). For a semigroup S, let I and J be two index sets, and let $P = (p_{ji})_{j\in J,\, i\in I}$ be a $J \times I$ matrix with entries from S. The set $I \times S \times J = \{(i, s, j)\,; i \in I, s \in S, j \in J\}$ with a multiplication

$$(i, s, j)(k, t, l) = (i, sp_{jk}t, l) \tag{1}$$

is defined as the Rees matrix semigroup denoted by $M_R = M[S; I, J; P]$. We should note that one can replace S by a semigroup with zero $S^0 = S \cup \{0\}$, or replace it by a group with zero $G^0 = G \cup \{0\}$ which is actually a semigroup (cf. [10]).

On the other hand, a semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero which means that the matrix P (in M_R) has a non-zero entry in each row and column. Briefly, for a semigroup G^0 (not necessarily same with S), a regular Rees matrix semigroup M_R constructed on G^0 defines a semigroup $M_C = M^0[G^0; I, J; P']$ with the same operation as (1) such that I and J are two index sets, and $P' = (p'_{ji})_{j\in J,\, i\in I}$ is a $J \times I$ matrix with entries G^0. Note that if the Rees matrix semigroup over a group is not regular, then we obtain a completely simple semigroup. We may refer [7, 10] to the reader about completely (0-)simple semigroups and some results on them.

In this paper, by considering notion of both Rees matrix semigroups and completely 0-simple semigroups, and then combining both of them, we will construct a *new semigroup structure* and classify it in the literature and study some properties on it. In this construction, we actually inspired by Howie's said which was *the importance of Rees recipe lies in its universality: every completely* 0*-simle semigroup is isomorphic to some Rees matrix semigroups.*

The notations and expressions used in each section are described in themselves. But two definitions to be used in the paper, especially second part, are as follows.

Definition 1.1 ([21]). A semigroup S is a *cryptic* if $\mathcal{H}$ is a congruence.

Definition 1.2 ([10]). Let S be a semigroup. A relation $\boldsymbol{R}$ on the set S is called left compatible if

$$(\forall\, s,t,v \in S)(s,t) \in \boldsymbol{R} \Rightarrow (vs, vt) \in \boldsymbol{R}$$

and right compatible if

$$(\forall\, s,t,v \in S)(s,t) \in \boldsymbol{R} \Rightarrow (sv, tv) \in \boldsymbol{R}.$$

Both left compatible and right compatible is called compatible. A compatible equivalence relation is called a *congruence*.

2. A New Semigroup $\mathcal{N}$ and Some Fundamental Results on it

As we mentioned in the previous section, we will construct a new semigroup based on Rees matrix and completely 0-simple semigroups. In the light of this thought, it will be of course so important to use the operation in (1) during to define a new operation for our new semigroup. In here, after obtaining the new construction of semigroups, we will further present some fundamental results on it to strengthen the theory.

Suppose that the Rees matrix semigroup $M^0[S^0; I, J; P]$ which was defined on the set $I \times S^0 \times J$ and completely 0-simple semigroup $M^0[G^0; I, J; P']$ which was defined on the set $I \times G^0 \times J$ are denoted by the notations M_R and M_C, respectively. For arbitrary elements $(a,b,c),(k,l,m) \in M_R$ and $(d,e,f),(x,y,z) \in M_C$, let us consider the mapping $\gamma : (M_R \times M_C) \star (M_R \times M_C) \to (M_R \times M_C)$ having binary operation $\star$ as

$$[(a,b,c),(d,e,f)] \star [(k,l,m),(x,y,z)]$$
$$= \begin{cases} ((a, bp_{ck}l, m), 0) & \text{if } p_{ck} \neq 0 \text{ and } p'_{fx} = 0, \\ (0, (d, ep'_{fx}y, z)) & \text{if } p_{ck} = 0 \text{ and } p'_{fx} \neq 0, \\ ((a, bp_{ck}l, m), (d, ep'_{fx}y, z)) & \text{if } p_{ck} \neq 0 \text{ and } p'_{fx} \neq 0, \\ (0_R, 0_C) & \text{if } p_{ck} = 0 \text{ and } p'_{fx} = 0. \end{cases} \quad (2)$$

Remark 2.1. A careful observation on (2) shows that the first line corresponds to the Rees matrix semigroup, the second line corresponds to the completely 0-simple semigroup. On the other hand the third line defines the general situation. The reason of this is although the operation in (2) a generalization for both these semigroups, it contains those inside of it as well.

Therefore, by taking into account the operation defined in (2) on the set $M_R \times M_C$, we have the following first theorem.

Theorem 2.2. *The set $M_R \times M_C$ defines a semigroup $M^0[S^0, G^0; M_R, M_C; P, P']$ with the operation given in* (2). *Let us denote this new semigroup by $\mathcal{N}$.*

Proof. It suffices to prove closure (well-defined) and associative properties over (2).

For any two elements $s_1 = (a, bp_{ck}l, m)$ and $s_2 = (a', b'p_{c'k'}l', m')$ in M_R, and also any two elements $s_3 = (d, ep'_{fx}y, z)$, $s_4 = (d', e'p'_{f'x'}y', z')$ in M_C, let us consider the element

$$t_1 = [(s_1, s_2), (s_3, s_4)] \in \mathcal{N}.$$

Moreover, let us take any other similar element $t_2 = [(s'_1, s'_2), (s'_3, s'_4)] \in \mathcal{N}$. Then the image of $(t_1, t_2) \in \mathcal{N} \times \mathcal{N}$ under the mapping γ must be one of the cases as defined in (2) since s_1, s_2, s'_1, s'_2 could be zero element in M_R and s_3, s_4, s'_3, s'_4 could be zero element in M_C. This implies the closure property. On the other hand, by taking any elements $t_1, t_2, t_3 \in \mathcal{N}$ and considering the operation in (2), it is not hard to see that the associative property $(t_1 \star t_2) \star t_3 = t_1 \star (t_2 \star t_3)$ is satisfied.

Hence the result. □

Recall that an *idempotent element* of a semigroup S is defined as $s^2 = s$ for any $s \in S$, and if for every $s \in S$ is idempotent then S is called the *idempotent semigroup* or *band*. Since the idempotent element and bands play important role in semigroup theory, our first fundamental result will be about whether $\mathcal{N}$ is an band or not.

The following lemma gives an explicit description of the idempotent element of $\mathcal{N}$.

Lemma 2.3. *The element* $[(a, b, c), (d, e, f)] \in \mathcal{N}$ *is an idempotent if and only if* $S \cup \{0\}$ *is a rectangular band and* $p'_{fd} = e^{-1}$.

Proof. By (2), the semigroup $\mathcal{N}$ consists of (in fact constructed by) M_R and M_C, and also a typical element of $\mathcal{N}$ can be taken as $t = [(a, b, c), (d, e, f)]$. Depends on choosing the elements, t has the following four possibilities.

$$\text{(i)} \quad [(a, 0, b), (d, e, f)], \quad \text{(ii)} \quad [(a, b, c), (d, 0, f)],$$
$$\text{(iii)} \quad [(a, b, c), (d, e, f)], \quad \text{(iv)} \quad [(a, 0, c), (d, 0, f)].$$

For the case (i),

$$[(a, 0, b), (d, e, f)] \star [(a, 0, b), (d, e, f)] = [(a, 0, b), (d, e, f)]$$
$$\Rightarrow [(a, 0, b), (d, ep_{fd}e, f)] = [(a, 0, b), (d, e, f)] \Rightarrow (d, ep_{fd}e, f) = (d, e, f)$$

which implies that $ep'_{fd}e = e$. Since case (i) falls into line 2 in the operation (2), e is an element of group. So $[(a, 0, b), (d, e, f)]$ is an idempotent element if $p'_{fd} = e^{-1}$. If a similar approximation apply to the case (ii), we have

$$[(a, b, c), (d, 0, f)] \star [(a, b, c), (d, 0, f)] = [(a, b, c), (d, 0, f)]$$
$$\Rightarrow [(a, bp_{ca}b, c), (d, 0, f)] = [(a, b, c), (d, 0, f)] \Rightarrow (a, bp_{ca}b, c) = (a, b, c)$$

which implies that $bp_{ca}b = b$. For each element of S it should be provided $bp_{ca}b = b$ if S (in here not S^0) is rectangular band then this equality holds. By a combination processing as in above cases, it is easy to see that cases (iii) and (iv) can be obtained in a similar manner.

Conversely, suppose that $S \cup \{0\}$ is rectangular band and $p'_{fd} = e^{-1}$. So the result follows by a direct calculation.

Hence the result. □

Lemma 2.4. *$\mathcal{N}$ is regular semigroup if and only if S^0 is regular semigroup.*

Proof. Let us consider that $\mathcal{N}$ is regular semigroup, i.e. for every $x \in \mathcal{N}$ there exists y such that xyx. $\mathcal{N}$ is of four different element forms by (2) and so we use the $[(a, b, c), (d, e, f)]$ element form that contains all of them.

$$[(a, b, c), (d, e, f)] \star [(k, l, m), (n, r, s)] \star [(a, b, c), (d, e, f)] = [(a, b, c), (d, e, f)]$$

$$[(a, bp_{ck}l, m), (d, ep'_{fn}r, s)] \star [(a, b, c), (d, e, f)]$$

$$= [(a, bp_{ck}lp_{ma}b, c), (d, ep'_{fn}rp'_{sd}e, f)]$$

$$[(a, bp_{ck}lp_{ma}b, c), (d, ep'_{fn}rp'_{sd}e, f)] = [(a, b, c), (d, e, f)].$$

We know that $p_{ck}lp_{ma}$ and $p'_{fn}rp'_{sd}$ are any element of S^0 and G^0, respectively. It means that we have two cases. The first of these $bp_{ck}lp_{ma}b = b$ other than $ep'_{fn}rp'_{sd}e = e$. Second condition is provided because we work on elements of G^0 and groups satisfy regular property. S^0 must be regular to hold the first case.

On the other hand suppose that S^0 is regular semigroup, $\mathcal{N}$ is clearly regular semigroup. □

We now discuss on *Green's relations* ([9]) which is very useful tool in the study of semigroups (and monoids). In particular, Green's relations can be used to classify given any semigroup (see, for instance, [1, 7, 10, 19]). By keeping this thought in our minds, at this part of the paper, we shall give our attention to the Green's relations on $\mathcal{N}$ which is the quite effective method to make a classification for a new semigroup. Now let us give a brief description of the Green's relations.

Let S be an arbitrary given semigroup. For any two elements, $a, b \in S$, the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ on S are defined as follows.

- $a\mathcal{L}b$ if and only if $S^1a = S^1b$.
- $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.
- $a\mathcal{H}b$ if and only if $a\mathcal{R}b$ and $a\mathcal{L}b$.

Each of Green's relations is an equivalence relation on the elements of S.

Proposition 2.5. *Let $[(a, b, c), (d, e, f)]$ and $[(k, l, m), (n, r, s)]$ be any two elements of the semigroup $\mathcal{N}$. Then the following hold.*

(i) $[(a, b, c), (d, e, f)]\mathcal{L}[(k, l, m), (n, r, s)] \Leftrightarrow c = m, f = s, b\,\mathcal{L}\,l$ *and* $e\,\mathcal{L}\,r$.

(ii) $[(a, b, c), (d, e, f)]\mathcal{R}[(k, l, m), (n, r, s)] \Leftrightarrow a = k, d = n, b\,\mathcal{R}\,l$ *and* $e\,\mathcal{R}\,r$.

(iii) $[(a,b,c),(d,e,f)]\mathcal{H}[(k,l,m),(n,r,s)] \Leftrightarrow a = k, d = n, c = m,\ f = s,\ b\,\mathcal{H}\,l$ *and* $e\,\mathcal{H}\,r$.

Proof. For the case (i), let $[(a,b,c),(d,e,f)]$, $[(k,l,m),(n,r,s)] \in \mathcal{N}$. By the definition of $\mathcal{L}$-Green's relations, if we have an element $[(k,y,z),(n,y',z')] \in \mathcal{N}$ such that

$$\begin{aligned}
&[(k,y,z),(n,y',z')] \star [(a,b,c),(d,e,f)] = [(k,l,m),(n,r,s)]\\
&\quad\Rightarrow [(k, yp_{za}b, c),(n, y'p'_{z'd}e, f)] = [(k,l,m),(n,r,s)]\\
&\quad\Rightarrow (k, yp_{za}b, c) = (k,l,m) \quad \text{and} \quad (n, y'p'_{z'd}e, f) = (n,r,s)\\
&\quad\Rightarrow yp_{za}b = l, \quad c = m, \quad y'p'_{z'd}e = r \quad \text{and} \quad f = s,
\end{aligned} \tag{3}$$

and also if we have an element $[(a,i,j),(d,i',j')] \in \mathcal{N}$ such that

$$\begin{aligned}
&[(a,i,j),(d,i',j')] \star [(k,l,m),(n,r,s)] = [(a,b,c),(d,e,f)]\\
&\quad\Rightarrow [(a, ip_{jk}l, m),(d, i'p'_{j'n}r, s)] = [(a,b,c),(d,e,f)]\\
&\quad\Rightarrow (a, ip_{jk}l, m) = (a,b,c) \quad \text{and} \quad (d, i'p'_{j'n}r, s) = (d,e,f)\\
&\quad\Rightarrow ip_{jk}l = b, \quad m = c, \quad i'p'_{j'n}r = e \quad \text{and} \quad s = f,
\end{aligned} \tag{4}$$

then, by (3) and (4), we obtain $b\,\mathcal{L}\,l$ and $e\,\mathcal{L}r$, as required.

Conversely, for any element $[(a,b,c),(d,e,f)], [(k,l,m),(n,r,s)] \in \mathcal{N}$, let us suppose that $c = m, f = s, b\mathcal{L}\,l$ and $e\mathcal{L}r$ are satisfied. Then we certainly get $[(a,b,c),(d,e,f)]\mathcal{L}[(k,l,m),(n,r,s)]$.

For the case (ii), we take two element $[(a,b,c),(d,e,f)]$, $[(k,l,m),(n,r,s)] \in \mathcal{N}$. By the definition of $\mathcal{R}$-Green's relations, if we have an element $[(x,y,m),(t,u,s)] \in \mathcal{N}$ such that

$$\begin{aligned}
&[(a,b,c),(d,e,f)] \star [(x,y,m),(t,u,s)] = [(k,l,m),(n,r,s)]\\
&\quad\Rightarrow [(a, bp_{cx}y, m),(d, ep'_{ft}u, s)] = [(k,l,m),(n,r,s)]\\
&\quad\Rightarrow (a, bp_{cx}y, m) = (k,l,m) \quad \text{and} \quad (d, ep'_{ft}u, s) = (n,r,s)\\
&\quad\Rightarrow bp_{cx}y = l, \quad a = k, \quad ep'_{ft}u = r \quad \text{and} \quad d = n,
\end{aligned} \tag{5}$$

and also if we have an element $[(x',y',c),(t',u',f)] \in \mathcal{N}$ such that

$$\begin{aligned}
&[(k,l,m),(n,r,s)] \star [(x',y',c),(t',u',f)] = [(a,b,c),(d,e,f)]\\
&\quad\Rightarrow [(k, lp_{mx'}y', c),(n, rp'_{st'}u', f)] = [(a,b,c),(d,e,f)]\\
&\quad\Rightarrow (k, lp_{mx'}y', c) = (a,b,c) \quad \text{and} \quad (n, rp'_{st'}u', f) = (d,e,f)\\
&\quad\Rightarrow lp_{mx'}y' = b, \quad k = a, \quad rp'_{st'}u' = e \quad \text{and} \quad n = d\,,
\end{aligned} \tag{6}$$

then, by (5) and (6), we obtain $b\,\mathcal{R}\,l$ and $e\,\mathcal{R}r$, as required.

Conversely, for any element $[(a,b,c),(d,e,f)]\,,[(k,l,m),(n,r,s)] \in \mathcal{N}$, let us suppose that $a=k$, $d=n$, $b\,\mathcal{R}\,l$ and $e\,\mathcal{R}\,r$ are satisfied. Then we certainly get $[(a,b,c),(d,e,f)]\mathcal{R}[(k,l,m),(n,r,s)]$.

The proof of (iii) is omitted since it is quite similar by considering the definition of $\mathcal{H}$-Green's relations. □

In this part, by using results on the idempotent element and Green relations, we will express the *inverse property* over the semigroup $\mathcal{N}$ to make another classification on it. Before giving the result, let us first consider the following lemma.

Lemma 2.6 ([14, Corollary 4.3]). *S is a completely inverse semigroup if and only if S is regular and $\mathcal{H}(S) = Z(E(S))$.*

Theorem 2.7. *$\mathcal{N}$ is completely inverse semigroup if and only if S^0 is regular commutative semigroup and the G^0 is commute.*

Proof. For the sufficiency part, let us assume that S^0 is regular commutative semigroup and the G^0 is commute. If we prove that $\mathcal{N}$ is regular and $\mathcal{H}(\mathcal{N}) = Z(E(\mathcal{N}))$ then $\mathcal{N}$ is completely inverse semigroup. By Lemma 2.4, since S^0 is regular semigroup then $\mathcal{N}$ is regular semigroup. Now if there exists an element $A \in \mathcal{Z}(E(\mathcal{N}))$ when $A \in \mathcal{H}(\mathcal{N})$ then $\mathcal{H}(\mathcal{N}) \subset \mathcal{Z}(E(\mathcal{N}))$. Let us consider A is of the form $[(x, bp_{zx}l, z), (i, ep'_{ki}r, k)]$. For idempotent element $[(x,y,z),(i,j,k)]$, we have

$$[(x, bp_{zx}l, z), (i, ep'_{ki}r, k)] \star [(x,y,z),(i,j,k)] = [(x, bp_{zx}lp_{zx}y, z), (i, ep'_{ki}rp'_{ki}j, k)].$$

Since $[(x,y,z),(i,j,k)]$ is idempotent element, by Lemma 2.3, we get $p'_{ki} = j^{-1}$. We also have $ep'_{ki}rp'_{ki}j = ej^{-1}r$. For the same idempotent element, we have

$$[(x,y,z),(i,j,k)] \star [(x, bp_{zx}l, z), (i, ep'_{ki}r, k)] = [(x, yp_{zx}bp_{zx}l, z), (i, jp'_{ki}ep'_{ki}r, k)].$$

Similarly, we get $jp'_{ki}ep'_{ki}r = ej^{-1}r$. Since S^0 is commute, the equivalence $bp_{zx}lp_{zx}y = yp_{zx}bp_{zx}l$ is satisfied. So we obtain $[(x, bp_{zx}lp_{zx}y, z), (i, ep'_{ki}rp'_{ki}j, k)] = [(x, yp_{zx}bp_{zx}l, z), (i, jp'_{ki}ep'_{ki}r, k)]$, which clearly implies $\mathcal{H}(\mathcal{N}) \subset \mathcal{Z}(E(\mathcal{N}))$.

Now we consider a dual argument, i.e. if there exist an element $B \in \mathcal{H}(\mathcal{N})$ when $B \in \mathcal{Z}(E(\mathcal{N}))$ then $\mathcal{Z}(E(\mathcal{N})) \subset \mathcal{H}(\mathcal{N})$. Let us take $B = [(x,\beta,z),(i,u,k)] \in Z(E(\mathcal{N}))$. Since $B = [(x,a,z),(i,b,k)] \star [(x,c,z),(i,d,k)]$ and $[(x,a,z),(i,b,k)], [(x,c,z),(i,d,k)] \in \mathcal{H}(\mathcal{N})$ we obtain $B \in \mathcal{H}(\mathcal{N})$, which clearly implies $\mathcal{Z}(E(\mathcal{N})) \subset \mathcal{H}(\mathcal{N})$. So $\mathcal{N}$ is completely inverse semigroup because it has been prove that $\mathcal{H}(\mathcal{N}) = Z(E(\mathcal{N}))$.

For the necessity part, let us assume that $\mathcal{N}$ is completely inverse semigroup. Thus, according to Lemma 2.6, $\mathcal{N}$ is regular and $\mathcal{H}(\mathcal{N}) = Z(E(\mathcal{N}))$. Firstly, since $\mathcal{N}$ needs to be regular, by Lemma 2.4, S is regular semigroup. As well as being $\mathcal{H}(\mathcal{N}) = Z(E(\mathcal{N}))$ it is seen that S^0 and G^0 are commute by direct calculations, as required.

Hence the result. □

Lemma 2.8 ([21, Lemma 1.4]). *The following conditions on a completely regular semigroup are equivalent.*

(1) *S is cryptic.*
(2) *S is a band of groups.*
(3) *S satisfies the identity $(ab)^0 = (a^0b^0)^0$.*
(4) *For any $a \in S$, $e \in E(S)$, $e < a^0$ implies that $ea = ae$.*

Lemma 2.9 ([10]). *A simple semigroup (without zero) is completely simple if and only if it is completely regular.*

Now our result is as follows.

Theorem 2.10. *If $\mathcal{N}$ is a completely regular semigroup then $\mathcal{N}$ is a band of groups.*

Proof. Firstly, we consider $S = 0$ and G has not zero element because the semigroup $\mathcal{N}$ is completely regular. This means that we work on elements of type $[(a, 0, b), (c, d, e)]$ by Lemma 2.9. Then it is must be shown that $\mathcal{N}$ is cryptic by Lemma 2.8.

In completely regular semigroup, let $s = [(a, 0, b), (c, d, e)]$ and $t = [(a', 0, b'), (c, f, e)]$ be any two elements of the semigroup $\mathcal{H}$-Green's relations. For $v = [(x, y, z), (k, l, m)] \in \mathcal{N}$ since

$$vs = [(x, 0, b), (k, lp'_{mc}d, e)]\mathcal{H}[(x, 0, b'), (k, lp'_{mc}f, e)] = vt$$

and

$$sv = [(a, 0, z), (c, dp'_{ek}l, m)]\mathcal{H}[(a', 0, z), (c, fp'_{ek}l, m)] = tv$$

such that $(s, t) \in \mathcal{H}$, $\mathcal{H}$ is a congruence by Definition 1.2 and so $\mathcal{N}$ is cryptic. By Lemma 2.8, $\mathcal{N}$ is a band of groups. □

3. Some Finiteness Conditions for $\mathcal{N}$

The study of finiteness conditions for semigroups consists in giving some conditions which are satisfied by finite semigroups and which are such as to assure the finiteness of them. For some studies in certain classes of semigroups and their constructions in terms of finiteness conditions, we may refer, for example, [2, 13, 17].

In this section, we will investigate of being finitely generated and of being periodicity for $\mathcal{N}$. Throughout this section G^0 and M will represent the 0-minimal ideal of S^0 and the set of all 0-minimal ideals of S^0, respectively.

3.1. *$\mathcal{N}$ is finitely generated*

In general, after obtaining some theoretical properties such as regularity, idempotent elements, Green's relations, etc., one may also try to find some other characterizations over the semigroup. One of the most economical way is to obtain the finite generating and relation sets. In this section, we therefore will establish the finitely generated (*f.g.*) property over this new semigroup $\mathcal{N}$. We may refer, for instance, the papers [2, 3, 18] to the readers for some related studies.

We note that although one of the next step of *f.g.* property is to obtain a finite presentation ([10]) for the semigroup that study on it, we will not come through the presentation of $\mathcal{N}$ in this paper and will leave it for future studies.

The following two lemmas will be needed for the result on *f.g.* property over $\mathcal{N}$.

Lemma 3.1. *If $\mathcal{N}$ is finitely generated, then I, J, $S^0\backslash G^0$ and M are finite sets.*

Proof. Suppose that $\mathcal{N}$ is a finitely generated semigroup. We should examine the proof according to the operation in Eq. (2).

(i) If G^0 is the zero element itself of 0-minimal ideal, then we clearly have the semigroup $\mathcal{N}$ constructed on the set $\mathcal{A} = [(I \times S^0\backslash G^0 \times J), 0]$ which coincides the result [3, Proposition 2.1]. In detail, the product of any two elements in the set $\mathcal{A}$ cannot give an element of $\mathcal{N}$, and so every element of the set $\mathcal{A}$ are indecomposable. According to [3], this gives that $S^0\backslash G^0$ is a finite set.

(ii) If $S^0\backslash G^0$ contains the zero element while M does not, then $\mathcal{N}$ is constructed on $\mathcal{B} = [0, (I \times M \times J)]$ which gives that $\mathcal{N}$ actually becomes a completely 0-simple semigroup, and so we can follow a similar technique as in the paper [3] to adapt this case. Therefore, if $\mathcal{N}$ has no zero element, then the minimal ideal G^0 of $\mathcal{N}$ is uniquely determined, but otherwise if $\mathcal{N}$ has a zero element. On the other hand, we can extend this idea to the set M which is clearly an ideal of $\mathcal{N}$. Let $M = \bigcup_{i=1}^{k} M_i$, where each of $M_1 = G^0, M_2, \ldots, M_k$ is a 0-minimal ideal of $\mathcal{N}$. Due to [20], each M_i $(1 \leq i \leq k)$ and so M is either a null semigroup or a 0-simple subsemigroup. To be a null semigroup of M gives the last condition of Eq. (2). Also, if M is a 0-simple subsemigroup, then the ideal M is uniquely determined up to again [20]. Hence every element in the set $I \times M \times J$ (or equivalently, in $\mathcal{B}$) are indecomposable, and so these indecomposable elements belong to the every generating set of $\mathcal{N}$. As in (i), this implies that M is finite.

(iii) If both $S^0\backslash G^0$ and M do not contain the zero element, then $\mathcal{N}$ is constructed on $[\mathcal{A}, \mathcal{B}]$ which coincides basically when $\mathcal{N}$ is constructed by both Rees factor and completely 0-simple semigroups as required in the general form. Therefore, by considering the cases (i) and (ii) at the same time, we reached the aimed.

As a result of these above progresses, if $\mathcal{N}$ is finitely generated, then each of I, J, S^0/G^0 and M is finite. $\square$

The next lemma describes a generating set for S^0 and G^0, seperately.

Lemma 3.2. *If X is generating set for $\mathcal{N}$, then the set $Y = A \cup \{p_{ji}; j \in J, i \in I\}$ generates S^0 and the set $Y' = A' \cup \{p'_{j'i'}; j' \in J, i' \in I\}$ generates G^0, respectively, where*

$$A = \{s \in S; ((i, s, j), 0) \in X\} \quad \textit{and} \quad A' = \{g \in G; (0, (i', g, j')) \in X\}.$$

Proof. By taking arbitrary elements $s \in S$; $i, k \in I$ and $j, l \in J$, we need to check decomposing for Y under the operation defined in (2).

$$\begin{aligned}((i,s,j),(k,0,l)) &= ((i_1,s_1,j_1),(k_1,0,l_1)) \star ((i_2,s_2,j_2),(k_2,0,l_2)) \\ &\quad \star \cdots \star ((i_m,s_m,j_m),(k_m,0,l_m)) \\ &= ((i_1,s_1p_{j_1i_2}s_2,j_2),(k_1,0,l_1)) \star \cdots \star ((i_m,s_m,j_m),(k_m,0,l_m)) \\ &= ((i_1,s_1p_{j_1i_2}s_2p_{j_2i_3}s_3\ldots p_{j_{m-1}i_m}s_m,j_m),(k_1,0,l_m)),\end{aligned}$$

and so, for $((i_1,s_1,j_1),(k_1,0,l_1)),\ldots,((i_m,s_m,j_m),(k_m,0,l_m)) \in X$, we conclude that

$$s = s_1p_{j_1i_2}s_2p_{j_2i_3}s_3\ldots p_{j_{m-1}i_m}s_m \in \langle Y\rangle \ .$$

Similarly, by considering arbitrary elements $g \in G$; $i', k' \in I$ and $j', l' \in J$, we have to check decomposing for Y' under the operation given in (2) since X contains a generating set for the completely zero simple semigroup as well.

$$\begin{aligned}((k',0,l'),(i',g,j')) &= ((k'_1,0,l'_1),(i'_1,g_1,j'_1)) \star ((k'_2,0,l'_2),(i'_2,g_2,j'_2)) \\ &\quad \star \cdots \star ((k'_m,0,l'_m),(i'_m,g_m,j'_m)) \\ &= ((k'_1,0,l'_2),(i'_1,g_1p_{j'_1i'_2}g_2,j'_2)) \star \cdots \star ((k'_m,0,l'_m),(i'_m,g_m,j'_m)) \\ &= (((k'_1,0,l'_m),(i'_1,g_1p_{j'_1i'_2}g_2p_{j'_2i'_3}g_3\ldots p_{j'_{m-1}i'_m}g_m,j'_m))),\end{aligned}$$

and then, for $((k'_1,0,l'_1),(i'_1,g_1,j'_1)),\ldots,((k'_m,0,l'_m),(i'_m,g_m,j'_m)) \in X$, we obtain

$$g = g_1p_{j'_1i'_2}g_2p_{j'_2i'_3}g_3\ldots p_{j'_{m-1}i'_m}g_m \in \langle Y'\rangle,$$

as required. □

Lemma 3.3 ([17, Theorem 1.1]). *Let T be a subsemigroup of a semigroup S. Then T is called large if the set $S\backslash T$ is finite. In addition, for a large subsemigroup T of S, we say that S is finitely generated if and only if T is finitely generated.*

Thus, the theorem on f.g. property for $\mathcal{N}$ can be indicated as follows.

Theorem 3.4. *Suppose that I and J are two (index) sets. Let S^0 be a semigroup with zero and $P = (p_{ji})_{j\in J, i\in I}$ be a $J \times I$ matrix over $S \cup \{0\}$. Also let G^0 be a group with zero and $P' = (p'_{ji})_{j\in J, i\in I}$ be a $J \times I$ matrix over $G \cup \{0\}$ such that every row or column of P' contains at least one non-zero entry. Then the semigroup $\mathcal{N}$ is finitely generated if and only if G^0 is finitely generated and I, J, $S^0\backslash G^0$ and M are all finite.*

Proof. *The necessity part:* Assume $\mathcal{N}$ is finitely generated. By Lemma 3.1, I, J, $S^0\backslash G^0$ and M must be finite. Also, by Lemma 3.2, the semigroup S^0 and the subsemigroup G^0 of it (by assumption G^0 is the 0-minimal ideal of S^0) are finitely generated since both Y and Y' are contained in X as generating set for $\mathcal{N}$.

The sufficiency part: Assume that the sets I, J, $S^0\backslash G^0$ and M are finite, and G^0 is finitely generated. By the assumption, since G^0 is subsemigroup of S^0 and $S^0\backslash G^0$ is finite, according to the Lemma 3.3, G^0 is large subsemigroup. So S^0 is finitely generated. Additionally, for the subset $C = S^0\backslash G^0$ of $\mathcal{N}$, it is easy to obtain that C is a subsemigroup of $\mathcal{N}$ by applying the operation defined in (2). Hence, since $\mathcal{N}\backslash C$ is finite while C is a subsemigroup of $\mathcal{N}$, then C is large by Lemma 3.3. Thus, again by Lemma 3.3, $\mathcal{N}$ is a finitely generated semigroup. □

3.2. *Periodicity of $\mathcal{N}$*

In this part, being periodicity for $\mathcal{N}$ will be investigated. The next lemma will be needed for the main result of this section.

Lemma 3.5. *If S and G are periodic, then $\mathcal{N}$ is periodic.*

Proof. Suppose that S and G are periodic. Then, by [2, Lemma 2.1], the Rees matrix semigroup $M[S; I, J; P]$ is also periodic. Now, by following a similar proof as in [2, Lemma 2.1], we will show that $\mathcal{N}$ is periodic.

For an arbitrary element $[(a, b, c), (d, e, f)] \in \mathcal{N}$, consider $bp_{ca} \in S$ and $ep_{fd} \in G$. So that there exist two different positive integers m and n such that

$$(bp_{ca})^m = (bp_{ca})^n \quad \text{and} \quad (ep_{fd})^m = (ep_{fd})^n.$$

It follows that

$$\begin{aligned}
[(a, b, c), (d, e, f)]^{m+1} &= [(a, b, c), (d, e, f)]^m[(a, b, c), (d, e, f)]^m \\
&= [(a, (bp_{ca})^m b, c), (d, (ep_{fd})^m e, f)] \\
&= [(a, (bp_{ca})^n b, c), (d, (ep_{fd})^n e, f)] \\
&= [(a, b, c), (d, e, f)]^n[(a, b, c), (d, e, f)]^n[(a, b, c), (d, e, f)]^{n+1}
\end{aligned}$$

which implies $\mathcal{N}$ is periodic, as required. □

Theorem 3.6. *$\mathcal{N}$ is periodic if and only if G^0 is periodic.*

Proof. (⇒:) Let us assume that $\mathcal{N}$ is a periodic semigroup. We know that $\mathcal{N}$ satisfies one of the four conditions in the operation defined in (2).

Firstly, let $\mathcal{N}$ be the type of $\mathcal{A} = [(I \times S^0\backslash G^0 \times J), 0]$ (or shortly, $(M_R, 0)$) which coincides with the case $\mathcal{N}$ is a Rees matrix semigroup. The periodicity of Rees matrix semigroups is clear by [2] which implies that G^0 have to be periodic.

Secondly, let $\mathcal{N}$ be the type of $\mathcal{B} = [0, (I \times M \times J)]$ (or shortly, $(0, M_C)$) which coincides with the case $\mathcal{N}$ is a completely 0-simple semigroup. With a similar way as in [2], let us consider an arbitrary element $bp_{ca}s$ of G^0 and then consider an element $[0, (a, sbp_{ca}sb, c)]$ of $\mathcal{N}$. There exist two different positive integers m and n

such that

$$\begin{aligned}&[0,(a, sbp'_{ca}sb, c)]^m\\&\quad= [0,(a, sbp'_{ca}sb, c)]^n\\&\quad\Rightarrow [0,(a,(sbp'_{ca})^{2m-1}sb, c)] = [0,(a,(sbp'_{ca})^{2n-1}sb, c)]\\&\quad\Rightarrow (sbp'_{ca})^{2m-1}sb = (sbp'_{ca})^{2n-1}sb \quad\text{or}\quad (sbp'_{ca})^{2m} = (sbp'_{ca})^{2n}\,.\end{aligned}$$

Therefore

$$\begin{aligned}(bp'_{ca}s)^{2m+1} &= (bp'_{ca}s)^{2m}(bp'_{ca}s) = bp'_{ca}(sbp'_{ca})^{2m}s\\&= bp'_{ca}(sbp'_{ca})^{2n}s = (bp'_{ca}s)^{2n+1},\end{aligned}$$

and thus it implies that G^0 is periodic.

Let $\mathcal{N}$ be the type of $[\mathcal{A},\mathcal{B}]$ (or, equivalently, (M_R, M_C)) which coincides the general case, in other words, $\mathcal{N}$ is constructed on both Rees matrix semigroup and completely 0-simple semigroup. Therefore, the above two paragraphs will give the solution of being G^0 is periodic.

It is clear that the remaining case in which $\mathcal{N}$ is the type of $(0_R, 0_C)$ is actually trivial.

($\Leftarrow$:) Suppose that G^0 is periodic. As we obtained in the previous section, G^0 is a large supsemigroup of S^0. Then, by [2, Theorem 5.1], S^0 is periodic since G^0 is periodic. Therefore, by Lemma 3.5, $\mathcal{N}$ is periodic since both S^0 and G^0 are periodic.

Hence the result. □

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